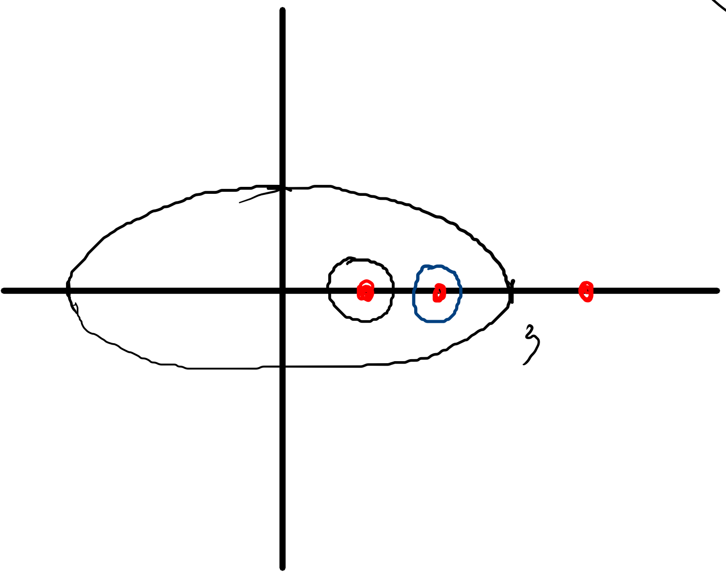
$$\int_\gamma \left(\frac{1}{z-1} + \frac{2}{z-2} + \frac{4}{z-4}\right) dz = 6\pi i$$

$f(z)$

$\gamma : [0, 2\pi] \to \mathbb{C}$ $\quad \gamma(t) = 3\cos t + i \,\mathrm{sen}\, t$

$\gamma(t) = 2 + \frac{1}{2}e^{it}$

$$\int_{C(2,\frac{1}{2})} \frac{2}{z-2}\, dz = \int_0^{2\pi} \frac{2}{\cancel{\frac{1}{2}e^{it}}} \cancel{\frac{1}{2}i e^{it}}\, dt = 4\pi i$$



$$= \underbrace{\int_\gamma \frac{1}{z-1}\, dz}_{=2\pi i} + \int_\gamma \frac{2}{z-2}\, dz + 4 \underbrace{\int_\gamma \frac{1}{z-4}\, dz}_{=0}$$

Cauchy

$$= \int_{C(1,\frac{1}{2})} \frac{1}{z-1}\, dz = \int_0^{2\pi} \frac{1}{\cancel{\frac{1}{2}e^{it}}} \cancel{\frac{1}{2}i e^{it}}\, dt = 2\pi i$$

Teorema dei 2 circuiti

$\gamma(t) = 1 + \frac{1}{2}e^{it}$

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n \qquad \boxed{c_{-1}} = \mathrm{Res}(f, z_0) \qquad c_{-2} \quad c_{-3} \; ?$$

Nel caso di un polo di ordine k come posso calcolare i coefficienti della serie di Laurent

$$\boxed{c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi}$$

$$f(z) = \frac{c_{-k}}{(z-z_0)^k} + \frac{c_{-k+1}}{(z-z_0)^{k-1}} + \dots + \frac{c_{-1}}{z-z_0} + c_0 + \dots$$

c_{-2} ?

$$g(z) = \underline{(z-z_0) f(z)}$$

$$g(z) = \frac{c_{-k}}{(z-z_0)^{k-1}} + \frac{c_{-k+1}}{(z-z_0)^{k-2}} + \dots + \boxed{\frac{c_{-2}}{z-z_0}} + c_{-1} + c_0 (z-z_0) + \dots$$

$$c_{-2} = \mathrm{Res}(g, z_0) = \frac{1}{(k-2)!} \lim_{z \to z_0} D^{k-2} \Big(\underbrace{(z-z_0)^{k-1} g(z)}_{(z-z_0)^k f(z)} \Big)$$

z_0 è polo di ordine $k-1$

$$c_{-3} = \mathrm{Res}\left((z-z_0)^2 f(z), z_0 \right) \dots$$

$$C_{-j} = \frac{1}{(k-j)!} \lim_{z \to z_0} D^{k-j} \left((z-z_0)^k f(z) \right).$$

$1 \leq j \leq k$

$$g(z) = (z-z_0)^k \sum_{n=k}^{+\infty} b_n (z-z_0)^{n-k}$$

<u>Oppure</u> $\frac{f(z)}{g(z)}$ f, g analitiche z_0 zero di g di molteplicità k $f(z_0) \neq 0$.

z_0 è polo di ordine k per f/g

$$\frac{f(z)}{g(z)} = \left[\frac{C_{-k}}{(z-z_0)^k} + \frac{C_{-k+1}}{(z-z_0)^{k-1}} + \dots + \frac{C_{-1}}{(z-z_0)} + C_0 + C_1(z-z_0) + \dots \right]$$

$$f(z) = \left[\quad \right] \cdot (z-z_0)^k \left[b_k + b_{k+1}(z-z_0) + b_{k+2}(z-z_0)^2 + \dots \right]$$

$$f(z) = \left[C_{-k} + C_{-k+1}(z-z_0) + \dots + C_{-1}(z-z_0)^{k-1} + C_0 (z-z_0)^k + \dots \right] \left[b_k + b_{k+1}(z-z_0) + \dots \right]$$

$$= d_0 + d_1(z-z_0) + d_2(z-z_0)^2 + \dots$$

$$d_0 = C_{-k} b_k \Rightarrow C_{-k} = \frac{d_0}{b_k}$$

$$d_1 = C_{-k} b_{k+1} + C_{-k+1} \cdot b_k \Rightarrow C_{-k+1} = \frac{d_1 - C_{-k} b_{k+1}}{b_k}$$

$f = \dfrac{1}{z - \operatorname{sen} z}$ $\quad z_0 = 0$ polo di ordine 3

$$\frac{\cancel{2}^{\cancel{2}} \cdot 3}{5 \cdot \cancel{4} \cdot \cancel{3} \cdot 2} = \frac{3}{10}$$

$$\operatorname{Res}(f, 0) = \frac{1}{2!} \lim_{z \to 0} D^2 \left(\frac{z^3}{z - \operatorname{sen} z} \right)$$

$$\frac{1}{z - \operatorname{sen} z} = \left[\frac{c_{-3}}{z^3} + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + \dots \right]$$

$$1 = [\dots\dots] \cdot (z - \operatorname{sen} z)$$

$$\operatorname{sen} z = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 \dots\dots$$

$$z - \operatorname{sen} z = \frac{1}{6} z^3 - \frac{1}{5!} z^5 + \frac{1}{7!} z^7 \dots$$

$$= z^3 \left(\frac{1}{6} - \frac{1}{5!} z^2 + \dots \right)$$

$$= (c_{-3} + c_{-2} z + c_{-1} z^2 + c_0 z^3 \dots) \left(\frac{1}{6} - \frac{1}{5!} z^2 + \dots \right)$$

$$1 = c_{-3} \cdot \frac{1}{6} \quad \rightsquigarrow \quad c_{-3} = 6$$

$$0 = c_{-3} \cdot \left(-\frac{1}{5!} \right) + c_{-1} \frac{1}{6} \quad \rightsquigarrow \quad c_{-3} = \frac{5!}{6} c_{-1} \qquad c_{-1} = \frac{6}{5!} \cdot 6$$

$$c_{-1} = \frac{3}{10}$$

$$0 = \frac{1}{6} c_{-2} z \quad \rightsquigarrow \quad c_{-2} = 0$$

$$\operatorname{Res}(f, 0) = \frac{3}{10} \qquad f(z) = \frac{6}{z^3} + \frac{3/10}{z} + \sum_{n=0}^{+\infty} c_n z^n \dots\dots$$

Funzioni razionali: decomposizione in frazioni semplici

$$f(z) = \frac{a(z)}{b(z)} = q(z) + \frac{r(z)}{b(z)} \leftarrow \deg r < \deg b$$

supponiamo $\deg a < \deg b$.

$$\boxed{b(z) = c\,(z-\alpha_1)^{k_1}\cdot(z-\alpha_2)^{k_2}\cdots(z-\alpha_n)^{k_n}}$$

$$\cancel{f(z) = \frac{c_1}{z-\alpha_1} + \frac{c_2}{z-\alpha_1} \cdots \frac{c_l}{z-\alpha_1} + D \cdots}$$

$$f(z) = \underbrace{\left[\frac{a_{1,1}}{z-\alpha_1} + \frac{a_{1,2}}{(z-\alpha_1)^2} + \cdots + \frac{a_{1,k_1}}{(z-\alpha_1)^{k_1}}\right]}_{\sigma_1(z)} + \underbrace{\left[\frac{a_{2,1}}{z-\alpha_2} + \frac{a_{2,2}}{(z-\alpha_2)^2} + \cdots + \frac{a_{2,k_2}}{(z-\alpha_2)^{k_2}}\right]}_{\sigma_2(z)} + \cdots$$

$$\cdots \underbrace{\left[\frac{a_{n,1}}{z-\alpha_n} + \frac{a_{n,2}}{(z-\alpha_n)^2} + \cdots + \frac{a_{n,k_n}}{(z-\alpha_n)^{k_n}}\right]}_{\sigma_n}$$

$\sigma_1(z)$ è la parte caratteristica della serie di Laurent di f in un intorno forato di α_1

Teorema

Sia $f(z) = \dfrac{a(z)}{b(z)}$ con a, b polinomi $\deg a < \deg b$. Detti $\alpha_1, \alpha_2, .., \alpha_n$ gli zeri di $b(z)$

si ha che $f(z) = \sum_{i=1}^{n} \sigma_i(z)$ dove $\sigma_i(z)$ è la parte caratteristica della serie di Laurent di f in un intorno forato di α_i.

Dim poniamo $g(z) = f(z) - \sum_{i=1}^{n} \sigma_i(z)$;

proviamo che $g(z)$ è intero: fuori da $\alpha_1, \alpha_2, .., \alpha_n$ è analitica; vediamo in α_j

scriviamo la serie di Laurent di $f(z)$ in un intorno forato di α_j

$$f(z) = \sigma_j(z) + \sum_{m=0}^{+\infty} c_m (z-\alpha_j)^m \qquad g(z) = \cancel{\sigma_j(z)} + \underbrace{\sum (c_m \,)}_{\text{regolare}} - \cancel{\sigma_j(z)} - \sum_{\substack{i=1 \\ i \neq j}} \sigma_i(z)$$

funzione analitica

$$\frac{1}{(z-\alpha_1)}\ldots$$

g è intero.

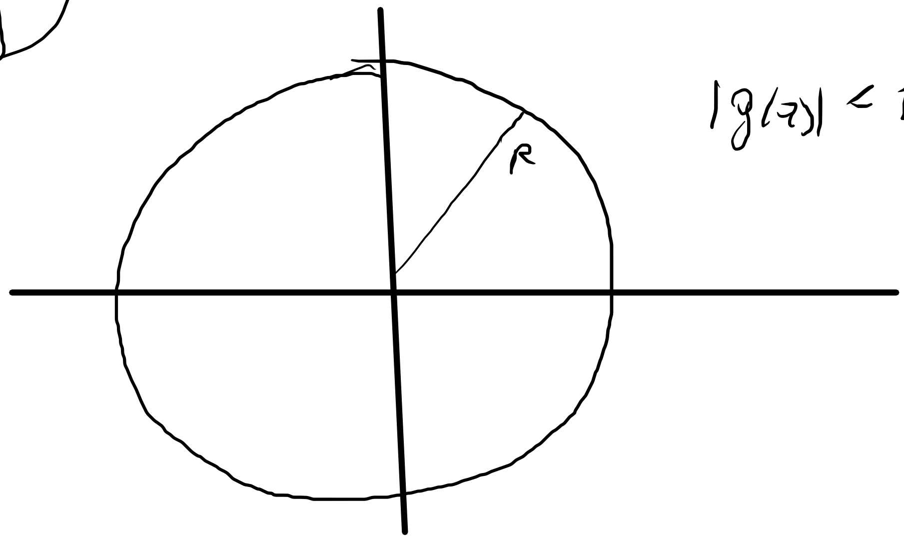
2) g è limitato

$$g(z) = f(z) - \sum_{i=1}^{n} \sigma_i(z)$$

$\deg a < \deg b$

$$\frac{a(z)}{b(z)} \to 0$$

$$\boxed{\lim_{|z| \to +\infty} g(z) = 0}$$



$|g(z)| < 1$ se $|z| > R$

Posso applicare il Teorema di Liouville $\Rightarrow$ $g(z) = \text{costante}$

$\Rightarrow g(z) = 0 \quad \forall z$ //

$$f(z) = \frac{z-1}{z^3(z+1)^2} = \left[\frac{\overset{-1}{\widehat{a_{1,3}}}}{z^3} + \frac{a_{1,2}}{z^2} + \frac{a_{1,1}}{z} \right] + \left[\frac{a_{2,2}}{(z+1)^2} + \frac{a_{2,1}}{z+1} \right]$$

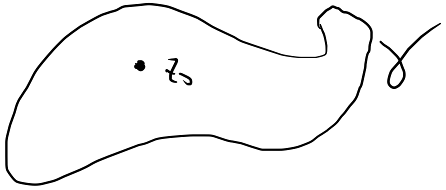
$$a_{1,1} = \mathrm{Res}(f, 0) \qquad a_{1,3} = \lim_{z \to 0} z^3 \cdot f(z) = \underline{-1}$$

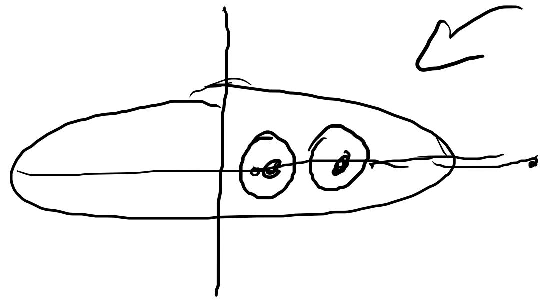
$k = 3$

$$\boxed{13.10}$$

Pause

$$\int_\gamma f\,dz = 2\pi i \,\mathrm{Res}(f, z_0)$$





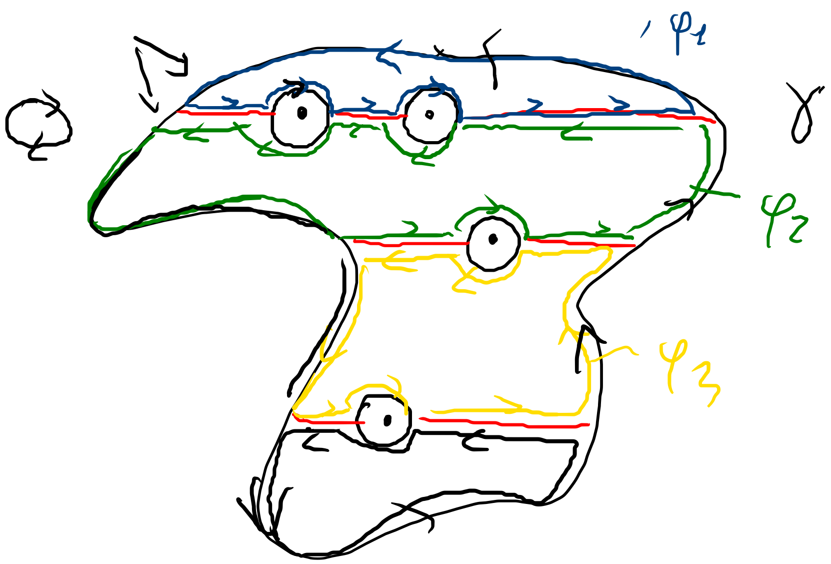
Teorema dei residui

$f: A \subseteq \mathbb{C} \to \mathbb{C}$ olomorfa, $z_1, z_2, \dots, z_n$ punti singolari isolati. γ $\Gamma \subset A$ e $\{z_1, \dots, z_n\} \subset \Gamma_{int}$ e non vi siano altre singolarità all'esterno di Γ.

Allora
$$\int_\gamma f\,dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i)$$

Dim

Dim



Per ogni z_i sia γ_i un cerchietto intorno a z_i contenuto in Γ_{int} e in modo tale che z_i sia l'unico punto singolare di f interno a γ_i.

$\Gamma_i \cap \Gamma_j = \emptyset$ se $i \neq j$

Troviamo delle rette parallele che passano per le singolarità e costruiamo dei circuiti $\varphi_1, \varphi_2, \dots, \varphi_m$

Per ogni φ_j $\quad \Phi_j \cup \Phi_{j\,int} \subset A$;

$$\int_{\varphi_j} f(z)\,dz = 0 \quad \text{per Cauchy}$$

$$\underbrace{\int_{\varphi_1} f + \int_{\varphi_2} f + \dots + \int_{\varphi_m} f}_{=0} = \underbrace{-\sum_{i=1}^{n} \int_{\gamma_i} f + \int_{\gamma} f}_{=0}$$

$$\Rightarrow \int_{\gamma} f\,dz = \sum_{i=1}^{n} \int_{\gamma_i} f\,dz = \sum_{i=1}^{n} 2\pi i \operatorname{Res}(f, \alpha_i)$$

Valor principale di una funzione

$f: \mathbb{R} \to \mathbb{R}$ $\qquad PV \int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R \to +\infty} \int_{-R}^{R} f(x)\,dx$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx$$

es: $f(x)$ dispari

$$\int_{-R}^{R} f(x)\,dx = 0 \quad \forall R!$$

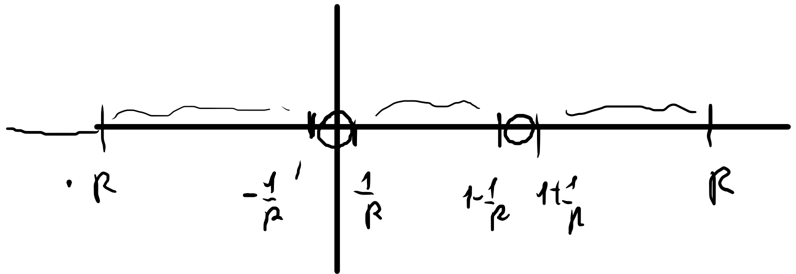
$$PV \int_{-\infty}^{+\infty} f(x)\,dx = 0$$

NB: se f è integrabile $PV\int = \int$.

f può presentare alcune singolarità

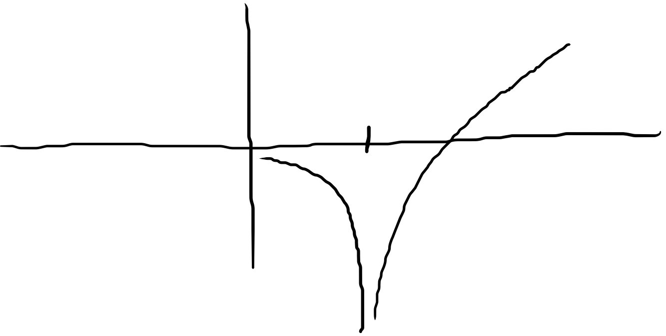
$$PV \int_{-\infty}^{+\infty} \frac{1}{x(x-1)}\,dx = \lim_{R \to +\infty} \left[\int_{-R}^{-\frac{1}{R}} f\,dx + \int_{\frac{1}{R}}^{1-\frac{1}{R}} f\,dx + \int_{1+\frac{1}{R}}^{R} f\,dx \right]$$

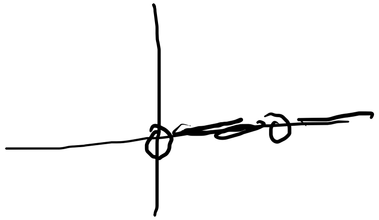
$x=0 \quad x=1!$



Es: Il logaritmo integrale

$$li(x) = PV\int_0^x \frac{1}{\log t}\,dt = \begin{cases} \lim\limits_{\varepsilon \to 0^+} \displaystyle\int_\varepsilon^x \frac{1}{\log t}\,dt & \text{se } x \in]0,1[\\ \lim\limits_{\varepsilon \to 0^+} \left[\displaystyle\int_\varepsilon^{1-\varepsilon} \frac{1}{\log t}\,dt + \int_{1+\varepsilon}^x \frac{1}{\log t}\,dt \right] \end{cases}$$

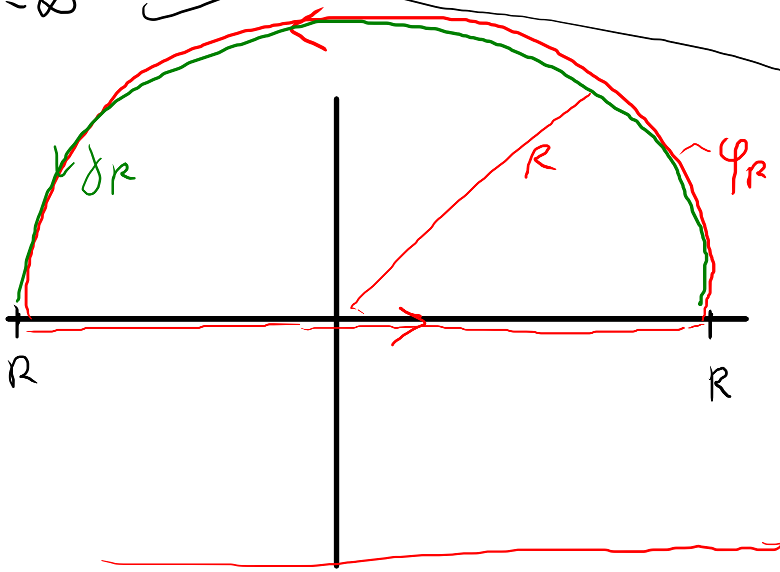




$\pi(x)$ = quanti sono i numeri primi $< x$

per x grandi $\pi(x) \approx li(x)$

Es: $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to+\infty}\int_{-R}^{R}\frac{1}{1+x^2}\,dx = \lim_{R\to+\infty}\left[\operatorname{arctg}(R)-\operatorname{arctg}(-R)\right] = \pi$



$$f(z) = \frac{1}{1+z^2}$$

sia $\varphi_R = \underline{[-R,R]} + \gamma_R$

$\gamma_R(t) = R e^{it} \quad t \in [0,\pi]$

Se provo che $\displaystyle\lim_{R\to+\infty}\int_{\gamma_R} f(z)\,dz = 0$ allora ho che $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to+\infty}\int_{\varphi_R} f\,dz$

$\Downarrow$

$\displaystyle\lim_{R\to+\infty}\int_{\varphi_R} f\,dz = \pi$

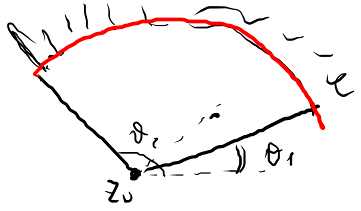
$$\int_{\varphi_R} f\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j)$$

$f(z) = \dfrac{1}{z^2+1}$ singolarità i, $-i$

$$\int_{\varphi_R} \frac{1}{1+z^2}\,dz = 2\pi i \operatorname{Res}\left(\frac{1}{1+z^2}, i\right) = 2\pi i \cdot \frac{1}{2i} = \pi$$

$$\frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$$

Lemma del cerchio grande

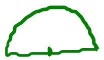


$f(z) = \frac{1}{z^2+1}$

$z_0 = 0 \quad R_0 > 1$

$\vartheta_1 = 0 \quad \vartheta_2 = \pi$

$\lambda = 0$



$S = S_{est}(z_0; R_0; \vartheta_1, \vartheta_2) =$

$= \{ z \in \mathbb{C} : |z - z_0| > R, \ \vartheta_1 \leq Arg(z - z_0) \leq \vartheta_2 \}$

$f : A \subseteq \mathbb{C} \to \mathbb{C}$ continua. $S \subseteq A$.

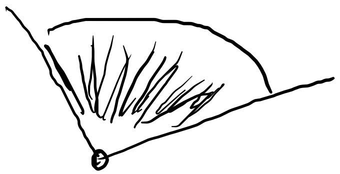
Esiste $\lim\limits_{\substack{|z| \to +\infty \\ z \in S}} (z - z_0) f(z) = \lambda \in \mathbb{C}$

$\lim\limits_{|z| \to +\infty} z \cdot \frac{1}{1+z^2} = 0$

Allora $\lim\limits_{R \to +\infty} \int_{\gamma_R} f(z)\, dz = \lambda i (\vartheta_2 - \vartheta_1)$

$\gamma_R : [\vartheta_1, \vartheta_2] \to \mathbb{C} \quad \gamma_R(t) = z_0 + R e^{it}$

Lemma del cerchio piccolo



$S = S_{int}(z_0; R_0, \vartheta_1, \vartheta_2) = \{ z \in \mathbb{C} : 0 < |z - z_0| < R_0, \ \vartheta_1 \leq Arg(z - z_0) \leq \vartheta_2 \}$

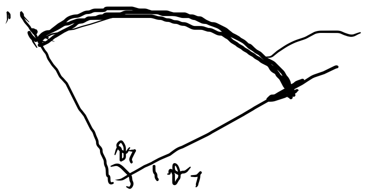
$S \subseteq A \qquad f : A \to \mathbb{C}$ continua.

Esiste $\lim\limits_{\substack{z \to 0 \\ z \in S}} (z - z_0) f(z) = \lambda \in \mathbb{C}$

Allora $\lim\limits_{R \to 0} \int_{\gamma_R} f(z)\, dz = \lambda i (\vartheta_2 - \vartheta_1)$

$\gamma_R : [\vartheta_1, \vartheta_2] \to \mathbb{C} \qquad \gamma_R(t) = z_0 + R e^{it}$

Dim   Poniamo $M_R = \max_{z\in\Gamma_R} \left|(z-z_0)f(z) - \lambda\right|$



γ_R

$\lambda \int_{\vartheta_1}^{\vartheta_2} \frac{1}{Re^{i\vartheta}}\, iRe^{i\vartheta}\, d\vartheta = \lambda i(\vartheta_2-\vartheta_1)$

Si ha (cerchio grande) $\lim_{R\to+\infty} M_R = 0$

(cerchio piccolo) $R\to 0$

$$\int_{\gamma_R} f(z)\,dz = \int_{\gamma_R}\left(f(z) - \frac{\lambda}{z-z_0} + \frac{\lambda}{z-z_0}\right)dz = \underbrace{\int_{\gamma_R}\frac{(z-z_0)f(z)-\lambda}{z-z_0}\,dz}_{*} + \lambda\int_{\gamma_R}\frac{1}{z-z_0}\,dz$$

proviamo che tende a 0 se $R\to+\infty$

$R\to 0$

$$|*| \le \int_{\gamma_R}\frac{M_R}{|z-z_0|}\,ds = \frac{M_R}{R}\overbrace{\int_{\gamma_R} ds}^{\text{lunghezza}} = \frac{M_R}{\cancel{R}}\cdot\cancel{R}\,(\vartheta_2-\vartheta_1) \to 0 \text{ se } R\to+\infty$$

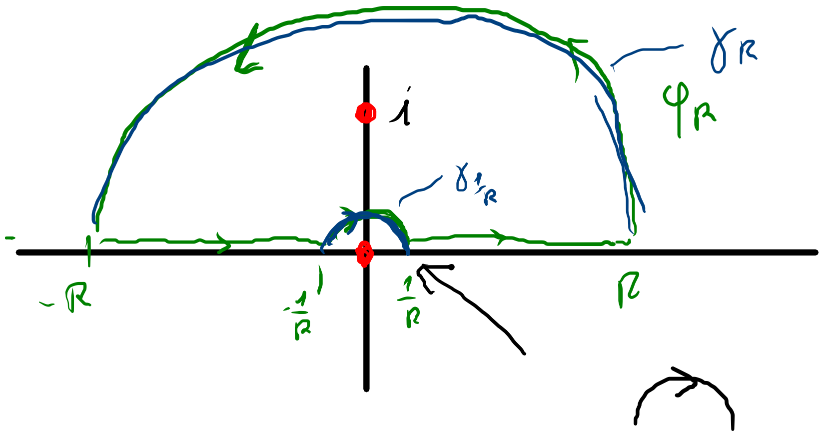
$|z-z_0| = R$, $z = z_0+Re^{it}$

$R\to 0$

$$PV \int_{-\infty}^{+\infty} \frac{1}{x(x-i)}\, dx$$

$$\uparrow$$

$$\lim_{R\to+\infty} \left(\int_{-R}^{-\frac{1}{R}} + \int_{\frac{1}{R}}^{R} \right)$$



$$\int_{\varphi_R} \frac{1}{z(z-i)}\, dz = 2\pi i \,\mathrm{Res}(f, i) = 2\pi i \cdot \frac{1}{i} = 2\pi$$

$$\int_{\varphi_R} = \boxed{\int_{-R}^{-1/R} \cdots + \int_{\frac{1}{R}}^{R}} - \int_{\gamma_{1/R}} + \int_{\gamma_R}$$

$$\|$$

$$2\pi$$

$\forall R > 1$

$$\lim_{R\to+\infty} [\ \ ] = \lim [\ \ ]$$

$$2\pi = PV\int_{-\infty}^{+\infty} - \lim_{\frac{1}{R}\to 0} \int_{\gamma_{1/R}} + \lim_{R\to+\infty} \int_{\gamma_R}$$