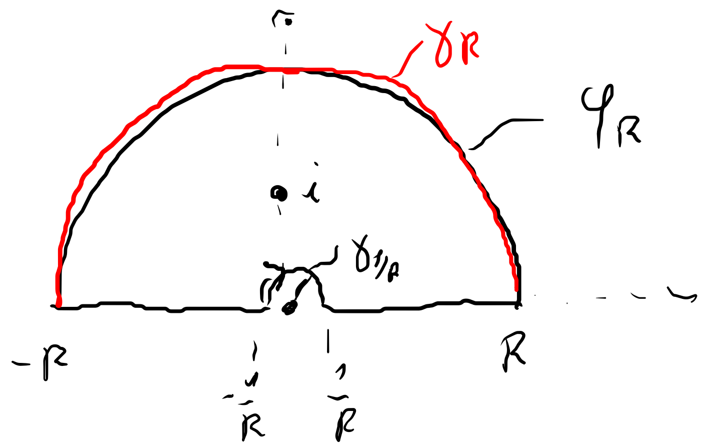
$$PV \int_{-\infty}^{+\infty} \frac{1}{x(x-i)}\, dx$$

$$f(z) = \frac{1}{z(z-i)}$$



$$\int_{\varphi_R} f(z)\, dz = 2\pi i \, \mathrm{Res}(f, i) = 2\pi$$

Lemma del cerchio grande $\lim_{|z|\to+\infty} z f(z) = 0$ $\quad \lim_{R\to+\infty} \int_{\gamma_R} f(z) dz = 0$

" " piccolo $\lim_{z\to 0} z \cdot \frac{1}{z(z-i)} = \frac{1}{-i} = i$

$$\lim_{R\to+\infty} \int_{\varphi_R} f(z) dz = 2\pi$$

$$\Rightarrow \lim_{\frac{1}{R}\to 0} \int_{\gamma_{1/R}} f(z) dz = i \cdot i\pi = -\pi$$

$$\lim_{R\to+\infty} \left[\underbrace{\left(\int_{-R}^{-1/R} f + \int_{1/R}^{R} f \right)} - \overbrace{\int_{\gamma_{1/R}} f}^{\pi} + \cancel{\int_{\gamma_R} f} \right] = 2\pi$$

$$\Rightarrow PV \int_{-\infty}^{+\infty} f(x) dx = 2\pi - \pi = \pi$$

Es:
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} e^{-i\omega x}\, dx$$

$\omega > 0$

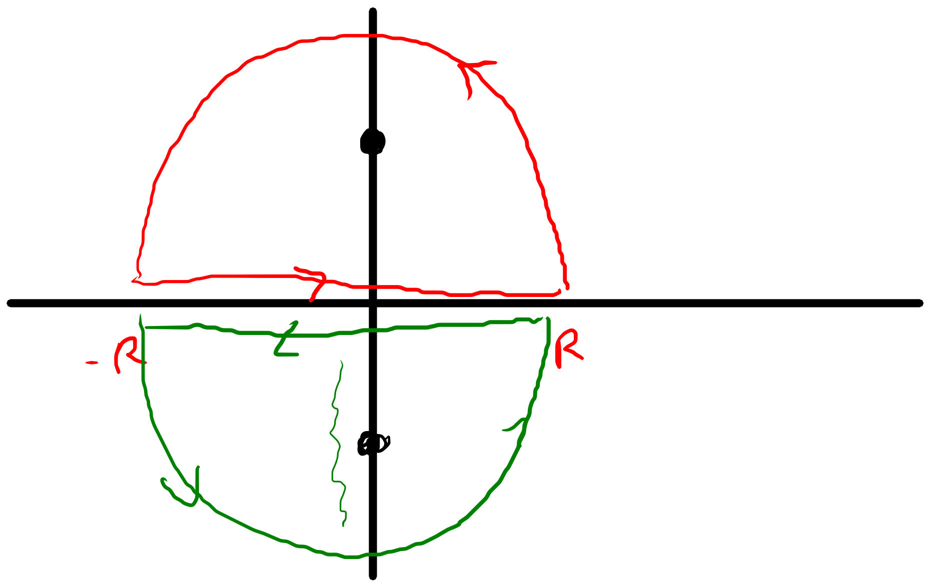
$$f(z) = \frac{1}{1+z^2} e^{-i\omega z}$$

$z = iy$

$e^{\omega y} \to +\infty$

$$\lim_{|z| \to +\infty} z f(z) = 0 \ ? \quad \text{Falso}$$

??



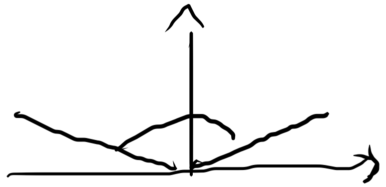
Lemma di Jordan

Sia $f : A \to \mathbb{C}$ continua, $S = S_{ext}(0 ; R_0 ; \vartheta_1, \vartheta_2) = \{ z \in \mathbb{C} : |z| > R_0, \ \vartheta_1 \le \operatorname{Arg} z \le \vartheta_2 \}$

$S \subseteq A$

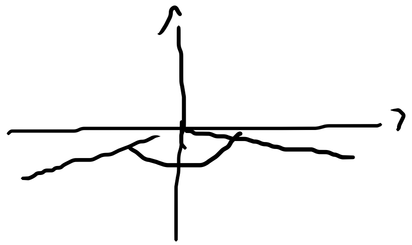
Sia $\displaystyle\lim_{\substack{|z| \to +\infty \\ z \in S}} f(z) = 0$. Sia $w \in \mathbb{R}$.

1) Sia $w > 0$ e $0 \le \vartheta_1 < \vartheta_2 \le \pi$



oppure

2) Sia $w < 0$ e $-\pi \le \vartheta_1 < \vartheta_2 \le 0$



Allora $\displaystyle\lim_{R \to +\infty} \int_{\gamma_R} e^{iwz} f(z) \, dz = 0$

$\gamma_R : [\vartheta_1, \vartheta_1] \to \mathbb{C}$

$\gamma_R(t) = R e^{it}$

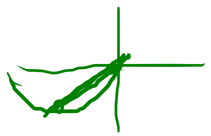
Dim Poniamo $M_R = \max_{z \in \Gamma_R} |f(z)|$ $\qquad \lim_{R \to +\infty} M_R = 0$

$w > 0 \quad \cancel{\ast} \quad w < 0$

$$\left| \int_{\gamma_R} f(z)\, e^{iwz}\, dz \right| \le \int_{\gamma_R} |f(z)|\, |e^{iwz}|\, ds \le \int_{\vartheta_1}^{\vartheta_2} M_R\, e^{-wR \operatorname{sen} t} \cdot R\, dt = (\ast)$$

$$z = R e^{it} = R(\cos t + i \operatorname{sen} t)$$

$$|e^{iwz}| = |e^{iwR(\cos t + i \operatorname{sen} t)}| = \underbrace{|e^{iwR\cos t}|}_{1} \cdot |e^{-wR \operatorname{sen} t}| = e^{-wR \operatorname{sen} t}$$



$$(\ast) \le \int_{\cancel{0}\,-\pi}^{\cancel{\pi}\,0} R M_R\, e^{-wR \operatorname{sen} t}\, dt =$$

$t \rightsquigarrow t + \pi = \tau \qquad \tau \in [0, \pi]$

$\operatorname{sen} t = \operatorname{sen}(\tau - \pi) = -\operatorname{sen}(\tau)$

$$\int_0^{\pi} R M_R\, e^{-|w| R \operatorname{sen}(\tau)}\, d\tau \to 0$$

$$= 2 R M_R \int_0^{\pi/2} e^{-wR \operatorname{sen} t}\, dt \le$$

$$\le 2 R M_R \int_0^{\pi/2} e^{-wR \frac{2}{\pi} t}\, dt = 2 R M_R \left[\frac{1}{-wR\frac{2}{\pi}}\, e^{-wR\frac{2}{\pi} t} \right]_0^{\pi/2} = \frac{2 \cancel{R} M_R}{w \cancel{R} \frac{2}{\pi}} \left(1 - e^{-wR}\right) = \frac{\pi}{w} M_R \left(1 - e^{-wR}\right)$$

$M_R \to 0$, $\quad e^{-wR} \to 0$, $\quad 1 - e^{-wR} \to 1$

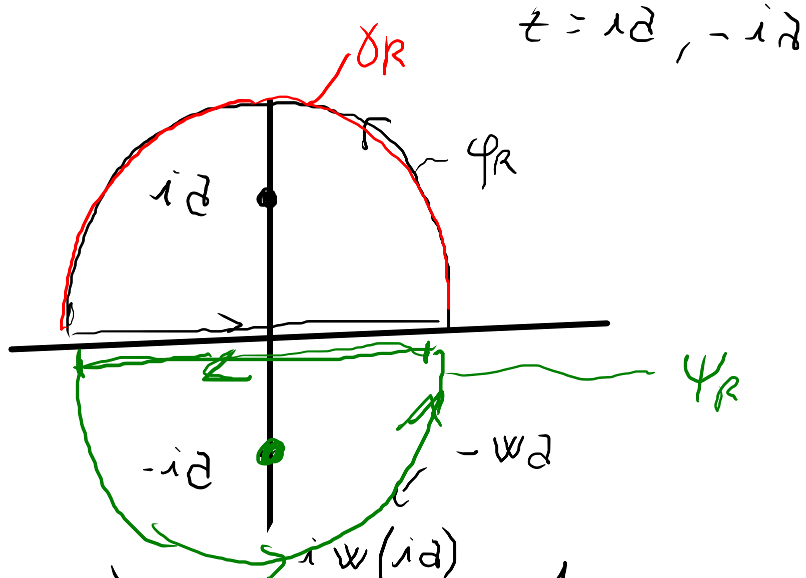
$\to 0$

$\mathcal{F}\left\{\frac{1}{a^2+x^2}\right\}$ $\quad a \in \mathbb{R}$ $\qquad W = -\omega$

$$\boxed{\lim_{|z|\to+\infty} \frac{1}{a^2+z^2} = 0}$$

$$\int_{-\infty}^{+\infty} e^{-i\omega x} \frac{1}{a^2+x^2}\,dx = \int_{-\infty}^{+\infty} e^{iWx} \frac{1}{a^2+x^2}\,dx$$

$z = ia\,,\ -ia$



$\omega > 0$ $\qquad W < 0$

se $\omega < 0$ quindi $W > 0$

$$\int_{\varphi_R\ \psi_R} e^{iWz} \frac{1}{a^2+z^2}\,dz = \boxed{2\pi i \operatorname{Res}(\ldots, ia)}$$

$$= -\int_{-R}^{R} e^{-i\omega x} \frac{1}{a^2+x^2}\,dx + \int_{\gamma_R} e^{iWz} \frac{1}{a^2+z^2}\,dz$$

$\searrow -\mathcal{F}\{f\}$ $\qquad \searrow 0$

$$\operatorname{Res}(\ ,ia) = e^{iW(ia)} \cdot \frac{1}{2ia} = \frac{1}{2i}\,\frac{1}{a}\,e^{-Wa}$$

$\cancel{2\pi i} \cdot [\ \ ]$

$$\Rightarrow -\mathcal{F}\{f\} = \frac{\pi}{a} e^{\omega a} \qquad \boxed{\omega < 0}$$

$$\operatorname{Res}(\ ,-ia) = e^{iW(-ia)} \cdot \frac{1}{-2ia} = -\frac{1}{2ia} e^{aW}$$

$$-\frac{\pi}{a} e^{-\omega a} \qquad \mathcal{F}\{f\} = \frac{\pi}{a} e^{-\omega a}$$

$$\mathcal{F}\{f\}(\omega) = \frac{\pi}{a} e^{-|\omega| a}$$

Se $\omega = 0$

$$\int_{-R}^{R} \cancel{e^{-i\omega x}} \frac{1}{a^2+x^2}\,dx = \int_{-R}^{R} \frac{1}{a^2} \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx$$

$\frac{x}{a} = u$

$du = \frac{1}{a}\,dx$

$$= \frac{1}{a}\pi$$

$$\frac{\pi}{a} e^{-a|\omega|}$$

$$\boxed{12.10}$$

seno cardinale

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\operatorname{sen}(\pi x)}{\pi x} & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases} \qquad \begin{cases} \dfrac{\operatorname{sen}(x)}{x} \\ 1 \end{cases}$$

seno integrale

$$Si(x) = \int_0^x \frac{\operatorname{sen} t}{t}\,dt \qquad \frac{\operatorname{sen} t}{t} \notin L^1(\mathbb{R})! \qquad \left|\frac{\operatorname{sen} t}{t}\right| \text{ non è integrabile}$$

è Riemann integrabile

$$\int_0^{+\infty} \underbrace{\frac{\operatorname{sen} x}{x}}_{\text{funz. pari}}\,dx = \lim_{x\to+\infty} Si(x)$$

$$PV\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx = \int_{-\infty}^{+\infty} \frac{\cos x}{x} + i\,\frac{\operatorname{sen} x}{x}$$

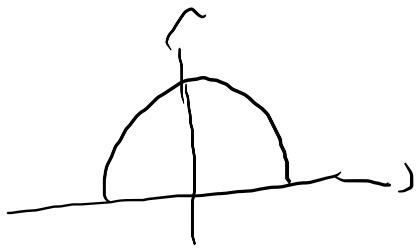
$\dfrac{\cos x}{x}$ è dispari $\Rightarrow PV\int = 0$

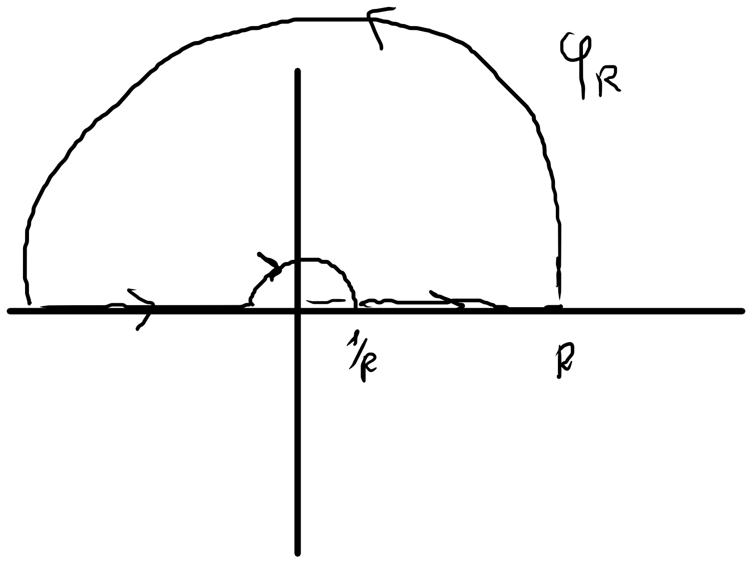
$$= \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\operatorname{sen} x}{x}\,dx = \frac{1}{2}\,\frac{1}{i}\,PV\int_{-\infty}^{+\infty} \frac{e^{ix}}{x}\,dx$$

$$PV\int_{-\infty}^{+\infty}\frac{e^{ix}}{x}\,dx$$

$f(z)=\frac{1}{z}$ $\qquad \lim_{|z|\to+\infty}\frac{1}{z}=0$

$e^{i(1)z}$ $\quad 1>0$ $\quad \cdot\frac{1}{z}$





φ_R

$1/R$ $\quad R$

$$\int_{\varphi_R}\frac{e^{iz}}{z}\,dz=0 \quad \text{(Cauchy)}$$

$$\left[\int_{-R}^{-1/R}\dots+\int_{1/R}^{R}\dots\right]-\underbrace{\int_{\gamma_{1/R}}}_{\to i\pi}+\underbrace{\int_{\gamma_R}\dots}_{\to 0}=0 \quad \forall R$$

Jordan $\Rightarrow \lim_{R\to+\infty}\int_{\gamma_R}e^{iz}\frac{1}{z}\,dz=0$

$\gamma_{1/R}(t)=\frac{1}{R}e^{it}$

$\gamma_R:[0,\pi]\to\mathbb{C}$

$\gamma_R(t)=R\,e^{it}$

Cerchio piccolo $\quad \lim_{z\to 0}e^{iz}\frac{1}{z}\cdot z=1$

$\Rightarrow \lim_{1/R\to 0}\int_{\gamma_{1/R}}e^{iz}\frac{1}{z}\,dz=1\cdot i\cdot\pi$

$$\lim_{R\to+\infty}\ \dots\ 0=PV\int_{-R}^{R}\frac{e^{ix}}{x}\,dx-i\pi$$

$$PV\int_{-\infty}^{+\infty}\frac{e^{ix}}{x}\,dx=i\pi$$

$$\boxed{\int_0^{+\infty}\frac{\operatorname{sen}x}{x}\,dx=\frac{\pi}{2}}\ //$$

Es.

$$PV\int_{-\infty}^{+\infty}\frac{x\,\mathrm{sen}\,x}{x^2+a^2}\,dx \qquad PV\int_{-\infty}^{+\infty}\frac{\cos x}{x^2+a^2}\,dx$$

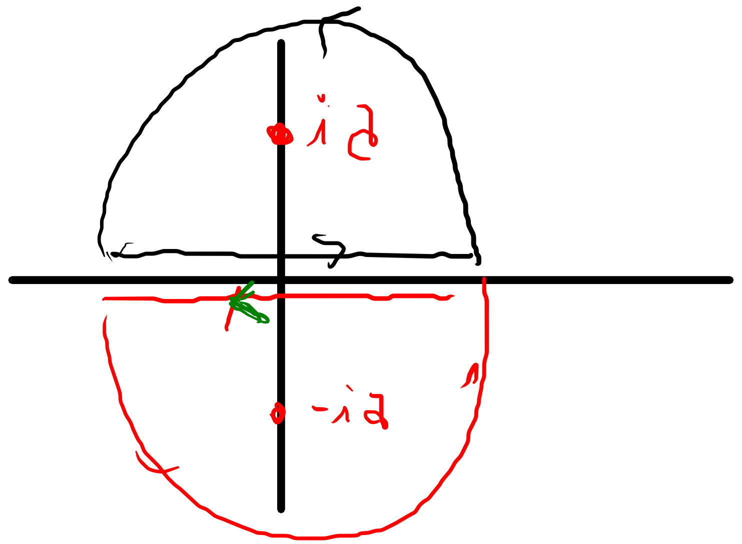
$$x\,\mathrm{sen}\,x=\frac{x}{2i}\left(e^{ix}-e^{-ix}\right) \qquad \cos x=\frac{1}{2}\left(e^{ix}+e^{-ix}\right)$$

$$\boxed{\frac{x\,e^{ix}}{x^2+a^2}} \qquad \boxed{\frac{x\,e^{-ix}}{x^2+a^2}}$$

$$PV\int_{-\infty}^{+\infty}\frac{x\,\mathrm{sen}\,x}{x^2+a^2}\,dx=\frac{1}{2i}\left[PV\int_{-\infty}^{+\infty}\frac{x\,e^{ix}}{x^2+a^2}\,dx-PV\int_{-\infty}^{+\infty}\frac{x\,e^{-ix}}{x^2+a^2}\,dx\right]$$

$$2\pi i\,\frac{e^{-a}}{2} \qquad -2\pi i\,\frac{e^{-a}}{2}$$

$$=\frac{1}{\cancel{2i}}\,\cancel{2}\pi\cancel{i}\,e^{-a}=\pi e^{-a}$$



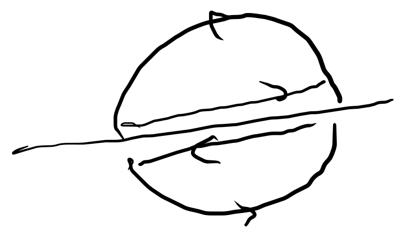
Jordan $\lim_{|z|\to+\infty}\frac{z}{z^2+a^2}=0$

$$\mathrm{Res}\left(\frac{z\,e^{iz}}{z^2+a^2},\,ia\right)=\frac{\cancel{ia}\,e^{i(ia)}}{2\cancel{ia}}=\frac{e^{-a}}{2}$$

$$\mathrm{Res}\left(\frac{z\,e^{-iz}}{z^2+a^2},\,-ia\right)=\frac{\cancel{-ia}\,e^{-i(-ia)}}{-2\cancel{ia}}=\frac{e^{-a}}{2}$$

$$PV \int_{-\infty}^{+\infty} \frac{\cos x}{x^2+a^2}\, dx = \frac{\pi}{a} e^{-a}$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$



$$\int \frac{e^{ix}}{x^2+a^2} \qquad \frac{e^{-ix}}{x^2+a^2}$$

$$\int_0^{2\pi} \frac{1}{1+\operatorname{sen}^2 t}\, dt$$

idea: trasformarlo in un integrale $\int_\gamma f(z)\,dz$

dove $\gamma(t) = e^{it}$ $\quad t \in [0, 2\pi]$

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} f\left(e^{it}\right) \cdot i\, e^{it}\, dt$$

Cerchiamo $f(z)$ tale che $f\left(e^{it}\right) \cdot i\, e^{it} = \frac{1}{1+\operatorname{sen}^2 t}$ $\rightarrow$ $f(z) = \frac{1}{iz} \cdot \left(\frac{1}{1+\operatorname{sen}^2 t}\right)$

$$\operatorname{sen} t = \frac{1}{2i}\left(e^{it} - e^{-it}\right) \qquad \frac{1}{1+\operatorname{sen}^2 t} \qquad f(z) \qquad z = e^{it}$$

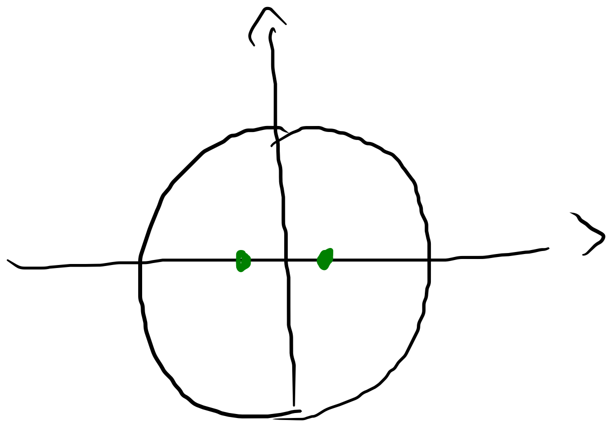
$$\operatorname{sen}^2 t = -\frac{1}{4}\left(e^{2it} - 2 + e^{-2it}\right)$$

$$\frac{1}{1+\operatorname{sen}^2 t} = \frac{1}{\frac{3}{2} - \frac{1}{4}e^{2it} - \frac{1}{4}e^{-2it}} = \frac{4}{6 - e^{2it} - e^{-2it}} \qquad z = e^{it}$$

$$= \frac{4}{6 - z^2 - z^{-2}} = \frac{4z^2}{6z^2 - z^4 - 1} = \frac{-4z^2}{z^4 - 6z^2 + 1}$$

$$f(z) = \frac{1}{iz} \cdot \frac{-4z^2}{z^4 - 6z^2 + 1} = \frac{4iz}{z^4 - 6z^2 + 1} \qquad \int_0^{2\pi} \frac{1}{1+\operatorname{sen}^2 t}\,dt = \int_\gamma \frac{4iz}{z^4 - 6z^2 + 1}\,dz$$

$$\int_\gamma \frac{-4iz}{z^4-6z^2+1}\,dz = 2\pi i \sum \mathrm{Res}(\ldots,\ )$$



singolarità: $u=z^2$ $\quad u^2-6u+1=0$

$3\pm\sqrt{9-1}$ $\quad u = 3\pm\sqrt{8} > 0$ $\quad z^2 = 3-\sqrt{8}$ oppure $z^2=3+\sqrt{8}$

$z=\pm\sqrt{3-\sqrt{8}}$ oppure $z=\pm\sqrt{3+\sqrt{8}}$

SI

non sono interne a Γ!

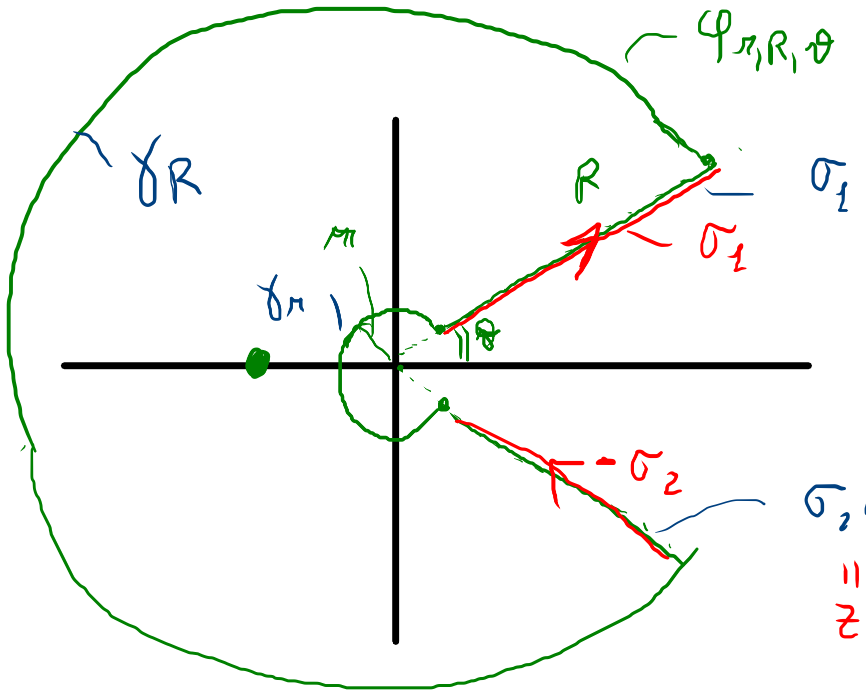
$$\mathrm{Res}\left(f,\sqrt{3-\sqrt{8}}\right) = \left[\frac{4iz}{4z^3-12z}\right]_{z=\sqrt{3-\sqrt{8}}} = \left.\frac{i}{z^2-3}\right|_{z=\sqrt{3-\sqrt{8}}} = \frac{i}{3-\sqrt{8}-3} = -\frac{i}{\sqrt{8}}$$

$$\mathrm{Res}\left(f,-\sqrt{3-\sqrt{8}}\right) = -\frac{i}{\sqrt{8}}$$

$$\to \int_0^{2\pi} \frac{1}{1+\sin^2 t}\,dt = 2\pi i\left(-\frac{2i}{\sqrt{8}}\right) = \frac{4\pi}{\sqrt{8}} = \sqrt{2}\,\pi$$

Esempio $\displaystyle\int_0^{+\infty} \frac{1}{x^{2/3}(1+x)}\,dx$

$\dfrac{1}{z^{2/3}(1+z)}$ non è analitica su $\mathbb{C}\setminus\{0,-1\}$



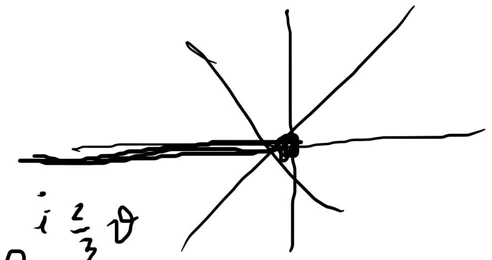
$\sigma_1(t) = t\,e^{i\vartheta}$

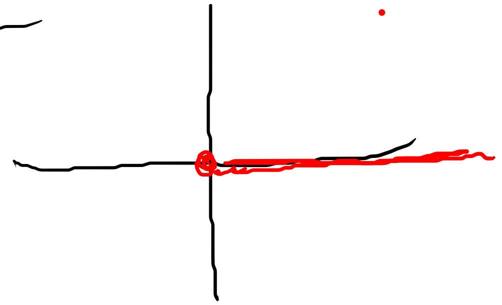
$t \in [r, R]$

$\sigma_2(t) = t\,e^{i(2\pi-\vartheta)}$

$z^{2/3} = (\rho e^{i\vartheta})^{2/3} = \rho^{2/3} e^{i\frac{2}{3}\vartheta}$

$\boxed{\vartheta \in\,]0, 2\pi[}$





$\overset{\shortparallel}{z} \quad z^{2/3} = t^{2/3}\cdot e^{\frac{2}{3}i(2\pi-\vartheta)}$

$R \to +\infty \quad r \to 0$

$\vartheta \to 0$

$$\int_{\sigma_2}\cdots = \int_r^R \frac{1}{t^{2/3}\cdot e^{\frac{2}{3}i(2\pi-\vartheta)}\cdot\left(1+t\,e^{i(2\pi-\vartheta)}\right)}\cdot e^{i(2\pi-\vartheta)}\,dt \longrightarrow \int_0^{+\infty}\frac{1}{t^{2/3}e^{\frac{4}{3}\pi i}(1+t)}\,dt = I\cdot e^{-\frac{4}{3}\pi i}$$

$$\int_{\sigma_1}\cdots = \int_r^R \frac{1}{t^{2/3}\,e^{\frac{2}{3}i\vartheta}(1+t\,e^{i\vartheta})}\cdot e^{i\vartheta}\,dt \longrightarrow \boxed{\int_0^{+\infty}\frac{1}{t^{2/3}(1+t)}\,dt} = I$$

$I\left(1 - e^{-\frac{4}{3}\pi i}\right) = \cdots$

$\dfrac{2}{\sqrt{3}}\pi$