$$\frac{d}{ds}F(s) = -\mathcal{L}\{t \cdot f(t)\}(s)$$

s^{-1} $\quad -1 \cdot s^{-2}$ $\quad (-1)(-2)s^{-3}$ ....

$f(t) = u(t)$ $\qquad \mathcal{L}\{t\,u(t)\}(s) = -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2}$

$$\mathcal{L}\{t^k u(t)\} = (-1)^k \frac{d^k}{ds^k}\frac{1}{s} = (-1)^k \cdot (-1)^k \cdot \frac{k!}{s^{k+1}}$$

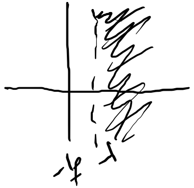
$$= \frac{k!}{s^{k+1}}$$

$$\frac{d^2}{ds^2}F(s) = \frac{d}{ds}\left(F'(s)\right) = +\mathcal{L}\{t^2 f(t)\}(s)$$

$$\underline{F^{(k)}(s) = (-1)^k \mathcal{L}\{t^k f(t)\}(s)}$$

Comportamento asintotico

Sia f trasformabile con ascissa di convergenza λ_f, sia $\lambda > \lambda_f$



allora F è limitata su $\bar{E}_\lambda = \{ s \in \mathbb{C} : Re(s) \geq \lambda \}$

(non su E_{λ_f} !) $\qquad f(t) = u(t) \quad \lambda_f = 0 \qquad F(s) = \frac{1}{s}$

Dim $s \in \bar{E}_\lambda$, allora $s = x + iy \quad x \geq \lambda$

$$|e^{-st} f(t)| \leq e^{-\lambda t} |f(t)| \qquad \underline{|F(s)|} = \left| \int_0^{+\infty} e^{-st} f(t)\,dt \right| \leq \int_0^{+\infty} e^{-\lambda t} |f(t)|\,dt = \underbrace{\mathcal{L}\{|f(t)|\}(\lambda)}_{\in \mathbb{R}}$$

Inoltre $\boxed{\lim_{Re(s) \to +\infty} F(s) = 0}$

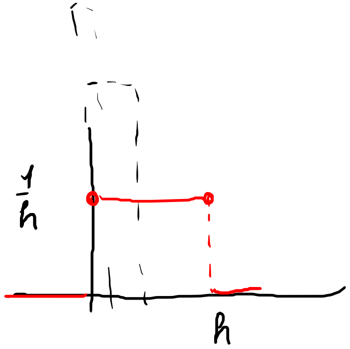
$$s_n \text{ t.c. } Re(s_n) \to +\infty \qquad \lim_{n \to +\infty} F(s_n) = \lim_{n \to +\infty} \int_0^{+\infty} \underbrace{e^{-s_n t} f(t)}_{f_n(t) = e^{-s_n t}} dt \overset{\text{Lebesgue}}{=} \int_0^{+\infty} \lim_{n \to +\infty} e^{-s_n t} f(t)\,dt =$$

$$= 0$$

$$|e^{-s_n t} f(t)| = \underset{\downarrow \atop 0}{e^{-Re(s_n) \cdot t}} |f(t)| \qquad |f_n(t)| \leq \underbrace{e^{-\lambda t} |f(t)|}_{\text{integrabile}}$$

se f è trasformabile $\lim_{Re(s) \to +\infty} F(s) = 0$

⚠ $\boxed{\text{Non è vero}}$ che $\lim_{|s| \to +\infty} F(s) = 0$



$\lambda_f = -\infty$

Es.: $f(t) = \underbrace{u(t) - u(t-1)} = \chi_{[0,1]}(t)$

$$F(s) = \int_0^{+\infty} \chi_{[0,1]}(t)\, e^{-st}\, dt = \int_0^1 e^{-st}\, dt = -\frac{1}{s}\left(e^{-s} - 1\right) = \begin{cases} \frac{1 - e^{-s}}{s} & s \neq 0 \\ 1 & s = 0 \end{cases}$$

$\lim_{Re(s) \to +\infty} F(s) = 0$

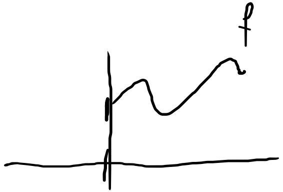
non esiste $\lim_{|s| \to +\infty} F(s)$

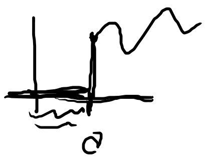
$\lim_{Re(s) \to -\infty} F(s) = -\infty$

Traslazione

$u(t) \overset{\mathcal{L}}{\rightsquigarrow} \frac{1}{s}$ $\qquad$ $u(t-1) \overset{\mathcal{L}}{\rightsquigarrow} e^{-s}\frac{1}{s}$

Sia f trasformabile, λ_f, $a \in \mathbb{R}$ $a>0$ $\qquad$ $g(t) = f(t-a)$





$f(t)u(t)$ $\qquad$ $g(t) = f(t-a)\,u(t-a)$

$$\mathcal{L}\{g\}(s) = \int_0^{+\infty} e^{-st} f(t-a)\,u(t-a)\,dt = \cancel{\int_{-a}^{+\infty} e^{-s(\tau+a)} f(\tau)\,u(\tau)\,d\tau}$$

$\tau = t-a$

$d\tau = dt$ $\qquad$ $t = \tau + a$

se $t=a \rightsquigarrow \tau = 0$

~~se $t=0$~~ $\rightsquigarrow \tau = -a$

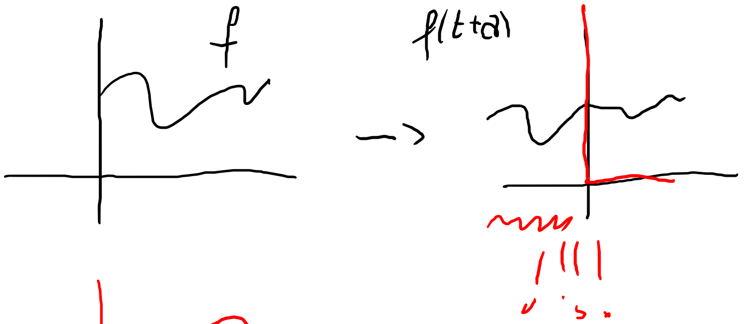
$$= \int_0^{+\infty} e^{-st} f(t-a)\,u(t-a)\,dt = \int_a^{+\infty} e^{-st} f(t-a)\,dt = \int_0^{+\infty} e^{-s(\tau+a)} f(\tau)\,d\tau = e^{-as}\,\mathcal{L}\{f\}(s)$$

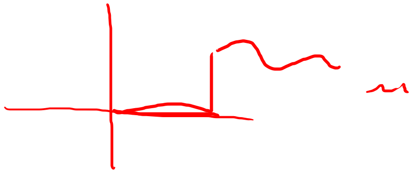
$u(t-a) = 0$ se $t<a$

$$\mathcal{L}\{f(t-a)\}(s) = e^{-as}\,\mathcal{L}\{f(t)\}(s)$$

$a>0$ $\quad f(t+a)$? $\quad \mathcal{L}\{f(t+a)\}(s) \overset{?}{=} e^{as}\,\mathcal{L}\{f(t)\}(s)$

?





Es si calcoli la trasformata di

$$f(t) = (t-3)^2\, u(t-3)$$

$$t^2 u(t) \rightsquigarrow \frac{2}{s^3} \qquad f(t) \rightsquigarrow e^{-3s}\frac{2}{s^3}$$

"Smorzamento"

f trasformabile, λ_f, $\quad g(t) = e^{\omega t} f(t) \qquad \omega \in \mathbb{C}$

$$G(s) = \int_0^{+\infty} e^{-st} \cdot e^{\omega t} f(t)\, dt = \int_0^{+\infty} e^{-(s-\omega)t} f(t)\, dt = F(s-\omega)$$

$$\lambda_g = \lambda_f + Re(\omega)$$

$Re(s-\omega) > \lambda_f$

$Re(s) > \lambda_f + Re(\omega)$

Es: $f(t) = u(t) \qquad F(s) = \frac{1}{s}$

$$g(t) = e^{\omega t} u(t) \rightsquigarrow G(s) = F(s-\omega) = \frac{1}{s-\omega}$$

$$\mathcal{L}\{e^{\omega t} u(t)\}(s) = \frac{1}{s-\omega}$$

$$\frac{s+\cancel{\omega}+s-\cancel{\omega}}{s^2-\omega^2}$$

$$f(t) = \cosh(\omega t) = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right) \overset{\mathcal{L}}{\rightsquigarrow} \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right) = \frac{s}{s^2-\omega^2}$$

$-$

$$f(t) = \operatorname{senh}(\omega t) = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right) \overset{\mathcal{L}}{\rightsquigarrow} \frac{\omega}{s^2-\omega^2}$$

$\cancel{s}+i\omega\cancel{-s}+i\omega$

$$f(t) = \operatorname{sen}(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) \overset{\mathcal{L}}{\rightsquigarrow} \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

$$f(t) = \cos(\omega t) = \frac{1}{2i}\left(\quad + \quad \right) \overset{\mathcal{L}}{\rightsquigarrow} \qquad = \frac{s}{s^2+\omega^2}$$

OSS $\quad \text{sen}(t) = \cos\left(t-\frac{\pi}{2}\right)$

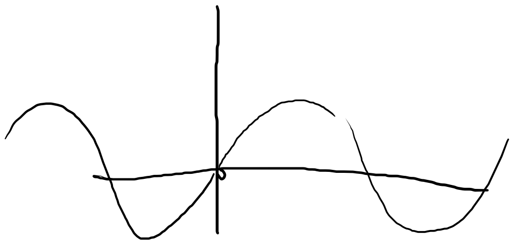
$$\mathcal{L}\{\cos t \overset{u(t)}{} \}(s) = \frac{s}{s^2+1}$$

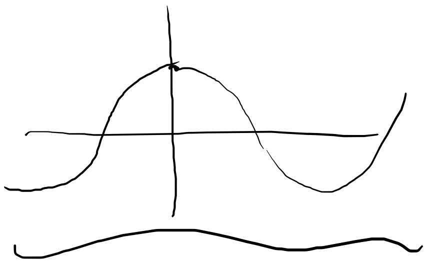
$$\mathcal{L}\{\cos(t-\tfrac{\pi}{2})\cdot u(t-\tfrac{\pi}{2})\}(s)$$

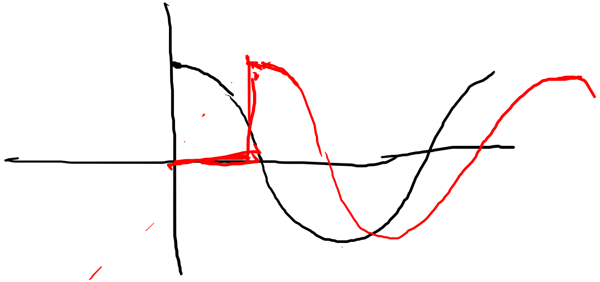
$$\mathcal{L}\{\text{sen}\, t\}(s) = \mathcal{L}\{\cos(t-\tfrac{\pi}{2})\}(s) = e^{-\frac{\pi}{2}s}\frac{s}{s^2+1} \overset{?}{=} \frac{1}{s^2+1}$$

$$\mathcal{L}\{\text{sen}\, t\cdot u(t)\} = \frac{1}{s^2+1}$$

NO







Riscalamento $\quad f(t) \rightsquigarrow F(s)$

$$g(t) = \overbrace{f(\omega t)} \qquad \omega \in \mathbb{R} \quad \boxed{\omega > 0}$$

$$-\frac{s}{\omega}t$$

$$G(s) = \int_0^{+\infty} e^{-st} f(\omega t)\,dt = \int_0^{+\infty} e^{-s\frac{\tau}{\omega}} f(\tau)\cdot\frac{1}{\omega}\,d\tau = \frac{1}{\omega}\mathcal{L}\{f(t)\}\left(\frac{s}{\omega}\right)$$

$\tau = \omega t \qquad t=0 \rightsquigarrow \tau=0$

$t = \frac{\tau}{\omega} \qquad t\to+\infty \rightsquigarrow \tau\to+\infty$

$d\tau = \omega\,dt$

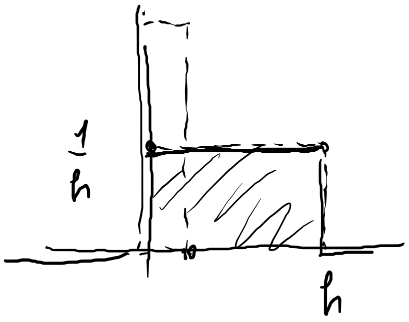
$$\mathrm{Re}\left(\frac{s}{\omega}\right) > \lambda_f$$

$$\mathcal{L}\{f(\omega t)\}(s) = \frac{1}{\omega}\mathcal{L}\{f(t)\}\left(\frac{s}{\omega}\right) \qquad \lambda_g = \omega\lambda_f$$

$$\operatorname{sen}(t) \rightsquigarrow \frac{1}{1+s^2} \qquad\qquad \frac{1}{\omega}F\left(\frac{s}{\omega}\right)$$

$$\operatorname{sen}(\omega t) \rightsquigarrow \frac{1}{\omega}\,\frac{1}{1+\left(\frac{s}{\omega}\right)^2} = \frac{1}{\omega}\,\frac{1}{\frac{\omega^2+s^2}{\omega^2}} = \frac{\omega}{\omega^2+s^2}$$

Impulso unitario



Distribuzione δ (delta) di Dirac

$h \in \mathbb{R} \quad h>0$

$$f_h(t) = \frac{1}{h}\,\chi_{[0,h]}(t)$$

impulso di durata h e altezza $\frac{1}{h}$

$$\int_{-\infty}^{+\infty} f_h(t)\,dt = 1 \quad \forall h$$

$$\lim_{h\to 0} u_h(t) = \begin{cases} 0 & \text{se } t \neq 0 \\ +\infty & \text{se } t = 0 \end{cases}$$

non è una funzione

$$f_h(t) = \frac{1}{h}\left(u(t) - u(t-h)\right)$$

$$u(t-h) \rightsquigarrow e^{-sh}\frac{1}{s}$$

$$\mathcal{L}\{f_h(t)\}(s) = \frac{1}{h}\left(\frac{1}{s} - \frac{e^{-sh}}{s}\right) = \begin{cases} \dfrac{e^{-sh}-1}{-sh} & \forall s \neq 0 \\ 1 & s = 0 \end{cases}$$

$\forall h \neq 0$

$$\lim_{h \to 0} \mathcal{L}\{f_h(t)\}(s) = 1 \qquad \forall s \in \mathbb{C}$$

Sia $\mathcal{D}$ uno spazio vettoriale di funzioni test (funzioni continue "belle")

$\varphi : \mathbb{R} \to \mathbb{R}$

Consideriamo per ogni h la funzione f_h e per ogni $\varphi \in \mathcal{D}$ il numero

$$\int_{-\infty}^{+\infty} \varphi(t) \cdot f_h(t)\, dt = \frac{1}{h} \int_0^h \varphi(t)\, dt$$

Se f è generico posso comunque considerare per ogni $\varphi \in \mathcal{D}$ il numero $\int_{-\infty}^{+\infty} \varphi(t) \cdot f(t)\, dt$

$\varphi \rightsquigarrow$ numero

Data f otteniamo una funzione $\boxed{\tilde{f} : \mathcal{D} \to \mathbb{C}}$ $\quad \tilde{f}(\varphi) = \int_{-\infty}^{+\infty} \varphi(t) f(t)\, dt$

Questo funzionale $\tilde{f}$ è una distribuzione, cioè una funzione da $\mathcal{D}$ a $\mathbb{C}$.

Per ogni $f: \mathbb{R} \to \mathbb{R}$ "ragionevole" posso considerare la distribuzione $\tilde{f}$.

Esistono però delle distribuzioni (cioè dei <u>funzionali</u> lineari definiti su $\mathcal{D}$ a valori in $\mathbb{C}$) che non sono del tipo $\tilde{f}$.

Ad esempio $\underline{\delta_0} : \mathcal{D} \to \mathbb{C}$ $\qquad \delta_0(\varphi) = \varphi(0)$ $\qquad$ distribuzione di Dirac centrata in 0

$$\left[\begin{array}{l} \tilde{f}(\alpha\varphi + \beta\psi) = \displaystyle\int_{-\infty}^{+\infty} \big(\alpha\varphi(t) + \beta\psi(t)\big) f(t)\,dt = \alpha \tilde{f}(\varphi) + \beta \tilde{f}(\psi) \quad ; \\ \delta_0(\alpha\varphi + \beta\psi) = \alpha\varphi(0) + \beta\psi(0) = \alpha\,\delta_0(\varphi) + \beta\,\delta_0\psi \end{array}\right]$$

OSSERVAZIONE

$\forall \varphi \in \mathcal{D}$

$$\lim_{h \to 0} \tilde{f}_h(\varphi) = \lim_{h\to 0} \int_{-\infty}^{+\infty} \varphi(t) \cdot f_h(t)\, dt = \lim_{h \to 0} \frac{1}{h} \int_0^h \varphi(t)\, dt = *$$

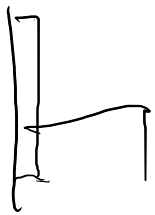
Teorema della media integrale

$$* = \lim_{h \to 0} \varphi(\xi_h) = \varphi(0) = \delta_0(\varphi)$$

$0 \leq \xi_h \leq h$

La famiglia di distribuzioni $\tilde{f}_h$ tende puntualmente a δ_0

δ_0 è "limite" delle $\tilde{f}_h$



$$\hat{f}_h \rightsquigarrow \delta_0$$

$\wr\wr$

f_h

$\wr$

$$\mathcal{L}\{f_h\} = \frac{e^{-hs}-1}{-hs} \rightsquigarrow 1$$

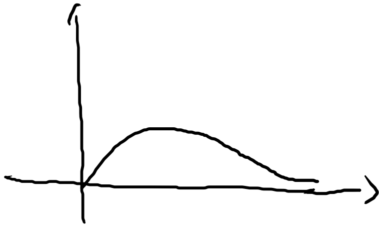
$$1 = \mathcal{L}\{\delta_0\}$$

⚠ NON VERIFICA $\lim_{Re(s)\to+\infty} F(s) = 0$

Funzione Γ di Eulero

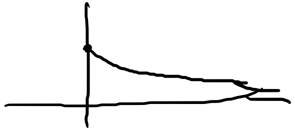
$x \in \mathbb{R}$ $\quad x>0 \qquad f:]0,+\infty[\to \mathbb{R} \qquad f(t) = t^{x-1} e^{-t}$

se $x>1$



$f(t)$ è integrabile su $]0,+\infty[$

se $x=1$

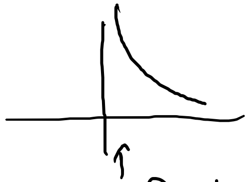


Definizione

$$\boxed{\Gamma(z) = \int_0^{+\infty} \underbrace{t^{z-1} e^{-t}}\,dt}$$

$Re(z) = x > 0 \qquad |t^{z-1} e^{-t}| = t^{x-1} e^{-t}$

se $0<x<1$



$\mathrm{Ord}_0 < 1$

$$\Gamma(1) = \int_0^{+\infty} e^{-t}\,dt = 1$$

$$\Gamma(z+1) = \int_0^{+\infty} t^z \underbrace{e^{-t}}_{e^{-t} = D(-e^{-t})}\,dt = \left[t^z(-e^{-t})\right]_0^{+\infty} - \int_0^{+\infty} z\, t^{z-1} \cdot (-e^{-t})\,dt = z\,\Gamma(z)$$

$$\boxed{\forall z \quad Re(z) > 0} \qquad \Gamma(z+1) = z \cdot \Gamma(z) \qquad \Gamma(1) = 1$$

$$\Gamma(2) = \Gamma(1+1) = 1 \cdot \Gamma(1) = 1 \qquad \Gamma(4) = 3 \cdot 2$$

$$\Gamma(3) = \Gamma(2+1) = 2 \cdot \Gamma(2) = 2 \qquad \Gamma(5) = 4 \cdot 3 \cdot 2$$

$$\Gamma(n+1) = n \cdot \Gamma(n) \qquad \Gamma(6) = 5 \cdot 4 \cdot 3 \cdot 2$$

$$\Gamma(n+1) = n!$$

OSS: possiamo estendere il dominio di Γ:

$$\Gamma(z+n) = (z+n-1) \cdot \Gamma(z+n-1) = (z+n-1)(z+n-2)\,\Gamma(z+n-2) =$$

$$= \underbrace{(z+n-1)(z+n-2)(z+n-3) \cdot \ldots \cdot z}\,\Gamma(z)$$

Sia $z \in \mathbb{C}$ prendo $n \in \mathbb{N}$ tale che $\mathrm{Re}(z+n) > 0$

allora posso definire $\Gamma(z+n)$

Possiamo definire $\Gamma(z)$ anche se $\mathrm{Re}(z) \not> 0$, ponendo

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\cdots(z+n-1)}$$

purché sia

$z \neq 0$

$z \neq -1$

$z \neq -2$

$\vdots$

$z \neq 1-n$

Γ si può prolungare a $\mathbb{C} \setminus \{0, -1, -2, -3, \dots\}$

Γ è analitica con singolarità isolate in $\mathbb{Z}^- \cup \{0\}$, che sono poli semplici.

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^{+\infty} t^{-1/2} e^{-t}\, dt = 2\int_0^{+\infty} e^{-\tau^2}\, d\tau = \sqrt{\pi}$$

$$\frac{1}{\sqrt{t}}$$

$$\sqrt{t} = \tau \qquad d\tau = \frac{1}{2\sqrt{t}}\, dt$$

$$t = 0 \rightsquigarrow \tau = 0$$

$$t \to +\infty \rightsquigarrow \tau \to +\infty$$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(1+\tfrac{1}{2}\right) = \frac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$f(t) = t^{\omega} u(t)$ $\quad \omega \in \mathbb{C}$ $\quad Re(\omega) > -1$ $\quad \omega = \alpha + i\beta$

$s = x + iy$

$$\int_0^{+\infty} \left| e^{-st} t^{\omega} \right| dt = \int_0^{+\infty} e^{-xt} t^{\alpha} \, dt =$$

$\tau = xt \quad d\tau = x\,dt$

$t = \frac{\tau}{x} \qquad t \to 0 \leadsto \tau \to 0 \qquad [x > 0]$

$t \to +\infty \leadsto \tau \to +\infty$

$$= \int_0^{+\infty} e^{-\tau} \left(\frac{\tau}{x}\right)^{\alpha} \cdot \frac{1}{x} \, d\tau = \frac{1}{x^{\alpha+1}} \int_0^{+\infty} e^{-\tau} \tau^{\alpha} \, d\tau = \quad \Gamma(\alpha+1)$$

$$= \frac{1}{x^{\alpha+1}} \int_0^{+\infty} \tau^{(\alpha+1-1)} e^{-\tau} \, d\tau =$$

$$= \frac{\Gamma(\alpha+1)}{x^{\alpha+1}}$$

Si ha $\forall$ $\alpha \in \mathbb{R}$ $\alpha > 0$ $\underbrace{\mathcal{L}\{t^\alpha u(t)\}}(x) = \dfrac{\Gamma(\alpha+1)}{x^{\alpha+1}}$. $x \in \mathbb{R}$ $x > 0$

$F(s)$ è analitica

$$F(x) = \frac{\Gamma(\alpha+1)}{x^{\alpha+1}} \quad \forall x \in \mathbb{R} \quad x > 0$$

la funzione $\dfrac{\Gamma(\alpha+1)}{x^{\alpha+1}}$ è prolungabile per analiticità su $\{s \in \mathbb{C} : Re(s) > 0\}$