$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$f(t) = t^{\alpha} u(t) \qquad \boxed{Re(\alpha) > -1} \qquad \lambda_f = 0 \qquad Re(s) > 0$$

$$\mathcal{L}\{t^{\alpha} u(t)\}(x) = \int_0^{+\infty} t^{\alpha} e^{-xt}\, dt = \int_0^{+\infty} \left(\frac{u}{x}\right)^{\alpha} e^{-u} \frac{1}{x}\, du = \frac{1}{x^{\alpha+1}} \int_0^{+\infty} u^{(\alpha+1)-1} e^{-u}\, du =$$

$x > 0$

$x = Re(s)$

$xt = u$ $\qquad t = \frac{u}{x}$

$du = x\, dt$

se $t \to 0$ $\quad u \to 0$

se $t \to +\infty$ $\quad u \to +\infty$

$$= \frac{1}{x^{\alpha+1}} \Gamma(\alpha+1)$$

considero il prolungamento analitico su $\{s \in \mathbb{C} : Re(s) > 0\}$ si ottiene

$$\mathcal{L}\{t^{\alpha} u(t)\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

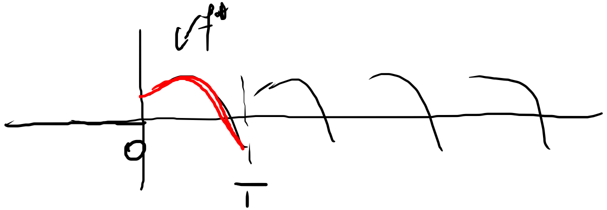
$\alpha = 1/2$ $\quad \mathcal{L}\{\sqrt{t}\, u(t)\}(s) = \dfrac{\Gamma(3/2)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2} s^{-3/2}$

$\alpha = -1/2$ $\quad \mathcal{L}\left\{\dfrac{1}{\sqrt{t}} u(t)\right\}(s) = \dfrac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$

$$\frac{1}{\sqrt{t}} \rightsquigarrow \sqrt{\pi}\,\frac{1}{\sqrt{s}}$$

$\alpha = n \in \mathbb{N}^+$ $\quad \mathcal{L}\{t^n u(t)\}(s) = \dfrac{\Gamma(n+1)}{s^{n+1}} = \dfrac{n!}{s^{n+1}}$

Segnali periodici



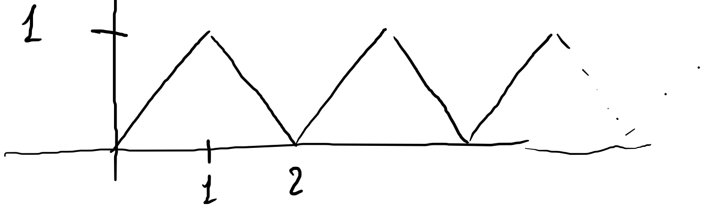
$f(t) \qquad f(t+T) = f(t) \quad \forall t \geq 0$

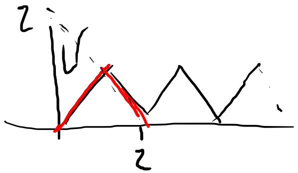
$$f^*(t) = f(t)\cdot\big(u(t) - u(t-T)\big)$$
$$= f(t)\,u(t) - \underbrace{f(t-T)}_{f(t)}\,u(t-T)$$

$$\mathcal{L}\{f^*\}(s) = F(s) - e^{-sT}F(s) = F(s)\left(1 - e^{-sT}\right)$$

$$\boxed{F(s) = \frac{\mathcal{L}\{f^*\}(s)}{1 - e^{-sT}}}$$

$T = 2$





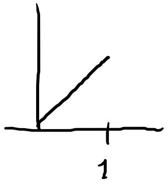
$$f^*(t) = \begin{cases} t & \text{si } t \in [0,1] \\ 2-t & \text{si } t \in [1,2] \\ 0 & \dots \end{cases} \qquad T = 2$$

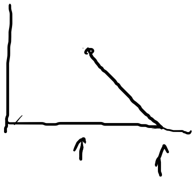
$-t\,u(t-1)$

$$f^*(t) = t[u(t) - u(t-1)] + (2-t)[u(t-1) - u(t-2)] = t\,u(t) - 2(t-1)\,u(t-1) + (t-2)\,u(t-2)$$

$-t\,u(t-1)$

$2\,u(t-1)$





$$\mathcal{L}\{f^*\}(s) = \frac{1}{s^2} - 2e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2} = \frac{(1-e^{-s})^2}{s^2}$$

$$\mathcal{L}\{f\}(s) = \frac{1}{\underbrace{1 - e^{-2s}}_{(1+e^{-s})(1-e^{-s})}} \frac{(1-e^{-s})^2}{s^2} = \frac{1 - e^{-s}}{s^2(1+e^{-s})}$$

Trasformata della funzione derivata

$f(t) = \frac{1}{\sqrt{t}} u(t)$ $\quad f'(t) = -\frac{1}{2} t^{-3/2} u(t)$ non è trasformabile

$t^{-1/2}$

<u>Teorema</u>

con ascissa di conv. λ_f

f segnale trasformabile, f derivabile su $]0,+\infty[$ (estendiamo $f'(t)$ anche per $t<0$)

<u>Sia f' trasformabile</u> con ascissa di convergenza $\lambda_{f'}$.

Supponiamo che esista finito $\boxed{\lim_{t \to 0^+} f(t) = f(0^+)}$

Allora si ha, per ogni s con $Re(s) > \max\{\lambda_f, \lambda_{f'}\}$

$$\boxed{\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0^+).}$$

Dim

per parti

$$\underbrace{\mathcal{L}\{f'\}(s)}_{\text{esiste}} = \int_0^{+\infty} e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_0^{+\infty} - \underbrace{\int_0^{+\infty} -s\, e^{-st} f(t)\,dt}$$

$$\lim_{t\to+\infty} e^{-st} f(t) - \underbrace{\lim_{t\to 0^+} e^{-st} f(t)}_{= f(0^+)}$$

?

$$\underbrace{f(0^+)}_{\text{esiste}}$$

$$+\,s \int_0^{+\infty} e^{-st} f(t)\,dt = \underbrace{s\,\mathcal{L}\{f\}(s)}_{\text{esiste}}$$

Supponiamo s con $Re(s) > \max\{\lambda_f, \lambda_{f'}\}$

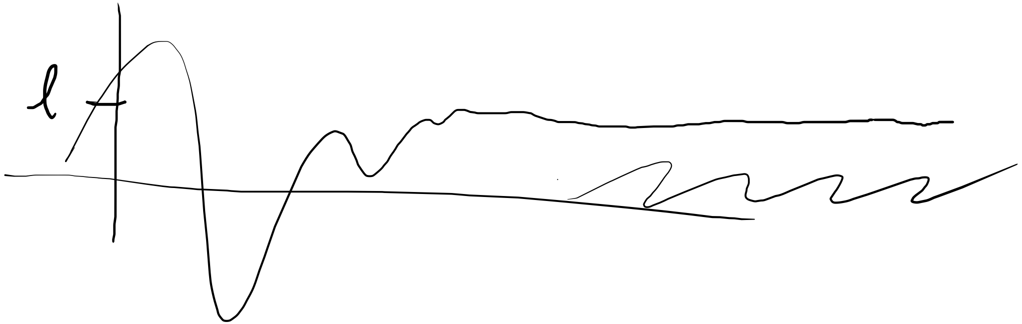
quindi esiste

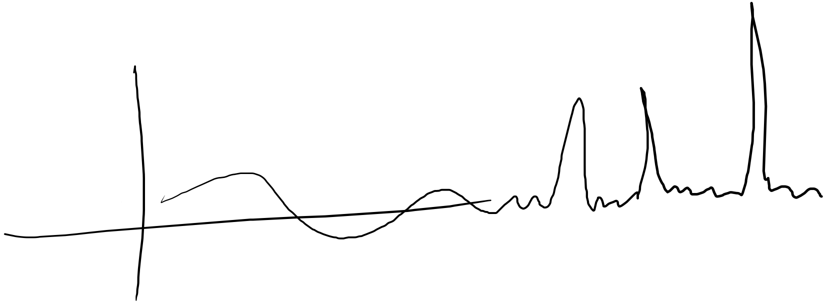
$$\lim_{t\to+\infty} e^{-st} f(t) = l$$

?

può essere $l \neq 0$?

La funzione $e^{-st} f(t)$ è integrabile su $[0,+\infty[$ perchè $Re(s) > \lambda_f$





quindi $\ell = 0$

$$\mathcal{L}\{f'(t)\}(s) = \underset{0}{\underset{\parallel}{\cancel{\lim_{t\to+\infty} e^{-st} f(t)}}} - f(0^+) + s\,\mathcal{L}\{f\}(s)$$

Es:
$$f(t) = \begin{cases} e^t & t \in]0,1[\\ 0 & \text{altrimenti} \end{cases} \qquad f'(t) = f(t)$$

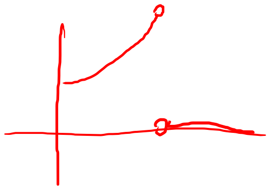
$$\mathcal{L}\{f\}(s) = \int_0^1 \underbrace{e^{-st}\, e^{t}}_{e^{t(1-s)}}\, dt = \left[\frac{1}{1-s}\, e^{t(1-s)}\right]_0^1 = \frac{1}{1-s}\left(e^{1-s} - 1\right)$$

$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0^+) = \frac{s}{1-s}\left(e^{1-s} - 1\right) - 1 = \frac{s\,e^{1-s} \cancel{- s} - 1 \cancel{+ s}}{1-s} =$$

$$\| \qquad \frac{s\,e^{1-s} - 1}{1-s}$$

$$\mathcal{L}\{f\}(s) = \frac{e^{1-s} - 1}{1-s} \;\overset{??}{\underset{??}{\neq}}\; \frac{s\,e^{1-s} - 1}{1-s}$$

?



Supponiamo che f abbia in $t_0 > 0$ un salto e per il resto sia derivabile su $]0,+\infty[$ allora si ha

$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - f(0^+) - \left(f(t_0^+) - f(t_0^-)\right) e^{-st_0}$$

$$\mathcal{L}\{f''\}(s) = s\,\mathcal{L}\{f'\}(s) - f'(0^+) - \left(f'(t_0^+) - f'(t_0^-)\right) e^{-st_0} =$$

$$= s\left[s\,\mathcal{L}\{f\}(s) - f(0^+) - \left(f(t_0^+) - f(t_0^-)\right) e^{-st_0}\right] - f'(0^+) - \left(f'(t_0^+) - f'(t_0^-)\right) e^{-st_0}$$

$$= s^2\,\mathcal{L}\{f\}(s) - s f(0^+) - f'(0^+) - s\left(f(t_0^+) - f(t_0^-)\right) e^{-st_0} - \left(f'(t_0^+) - f'(t_0^-)\right) e^{-st_0}$$

$$\mathcal{L}\{f^{(n)}\}(s) = s^n F(s) - s^{(n-1)} f(0^+) - s^{(n-2)} f'(0^+) - \dots - s f^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$

13.10

Prodotto di convoluzione

$f, g \in L^1(\mathbb{R}^N)$ $\quad f * g\,(x) = \int_{\mathbb{R}^N} f(y)\, g(x-y)\, dy$

$N = 1$ per quasi ogni x la funzione $f(y)\, g(x-y)$ è integrabile in y

Teorema

Siano $f, g \in L^1(\mathbb{R})$ allora per quasi ogni $x \in \mathbb{R}$ la funzione $f(\cdot)\, g(x - \cdot)$ è integrabile su $\mathbb{R}$.

Utilizziamo i teoremi di Fubini e Tonelli.

Teorema di Fubini (integrale di ~~Riemann~~ Lebesgue)

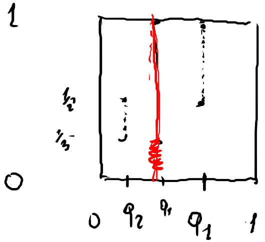
$\boxed{h : A \times B \to \mathbb{R}/\mathbb{C} \text{ integrabile}}$. Allora per ~~ogni $a \in A$~~ per quasi ogni $a \in A$ la funzione $h(a,\cdot)$ è integrabile su B e la funzione $\varphi : A \to \mathbb{R}$ $\varphi(a) = \int_B h(a,y)\,dy$,

allora $\int_A \left(\int_B h(a,y)\,dy \right) da = \iint_{A\times B} h(x,y)\,dx\,dy.$

⚠ Esempio. $\mathbb{Q} \cap [0,1] = \{ q_n : n \in \mathbb{N}^+ \}$

$$h(x,y) = \begin{cases} \text{se } x = q_n \text{ e } y \in \left] \frac{1}{n+1}, \frac{1}{n} \right[\cap \mathbb{Q} \text{ allora } h(x,y) = 1 \\ 0 \quad \text{altrimenti} \end{cases}$$

$h : [0,1]\times[0,1] \to \mathbb{R}$



$h(x,y)$ è integrabile su $[0,1]\times[0,1]$

$$\iint_{[0,1]\times[0,1]} h(x,y)\,dx\,dy = 0$$

$h(q_n, y)$ non è integrabile su $[0,1]$

Teorema di Tonelli

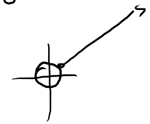
$h: \mathbb{R}^2 \to \mathbb{R}$ misurabile e $\boxed{h(x,y) \geq 0}$ q.o. in $\mathbb{R}^2$; supponiamo che per quasi ogni $y \in \mathbb{R}$ la restrizione $h(\cdot, y)$ sia integrabile su $\mathbb{R}$ e la funzione $\varphi(y) = \int_{\mathbb{R}} h(x,y)\,dx$ sia integrabile.

Allora h è integrabile su $\mathbb{R}^2$ (e quindi $\iint_{\mathbb{R}^2} h(x,y)\,dx\,dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x,y)\,dx \right) dy$)

Esempio $\quad h(x,y) = \begin{cases} \dfrac{x}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$\int_{-\infty}^{+\infty} |h(x,y)|\,dx$ è integrabile

$\forall y \neq 0 \quad h(x,y) = \dfrac{x}{x^4\left(1+\left(\frac{y}{x}\right)^2\right)^2} = \underbrace{\dfrac{1}{x^3\left(1+\left(\frac{y}{x}\right)^2\right)^2}} \sim \dfrac{1}{x^3}$ $\quad y \neq 0$



$\int_{\mathbb{R}} \dfrac{x}{(x^2+y^2)^2}\,dx = 0$ la funzione è dispari!

Però h non è integrabile su $\mathbb{R}^2$ $\quad \iint_{\mathbb{R}^2} |h(x,y)|\,dx\,dy = \lim_{\varepsilon \to 0} \iint_{\mathbb{R}^2 \setminus B(0,\varepsilon)} |h(x,y)|\,dx\,dy =$

$= \lim_{\varepsilon \to 0} \int_0^{2\pi} \left(\int_\varepsilon^{+\infty} \left| \dfrac{\rho \cos\vartheta}{\rho^4} \cdot \rho \right| d\rho \right) d\vartheta \leq \lim_{\varepsilon \to 0} 2\pi \int_\varepsilon^{+\infty} \dfrac{1}{\rho^2}\,d\rho = +\infty$

$\left[-\dfrac{1}{\rho} \right]_\varepsilon^{+\infty}$

Teorema $\quad f, g \in L^1(\mathbb{R}) \Rightarrow \exists \; f * g \in L^1(\mathbb{R})$

$$f * g(x) = \int_{\mathbb{R}} f(y)\, g(x-y)\, dy$$

$$h(x,y) = |f(y)\, g(x-y)| \geq 0$$

$h(\cdot, y) = |f(y)| \cdot |g(x-y)|$ è integrabile su $\mathbb{R}$ (in x)

$$\int_{\mathbb{R}} h \, dx = |f(y)| \underbrace{\int_{-\infty}^{+\infty} |g(x-y)|\, dx}_{u = x-y} = \int_{-\infty}^{+\infty} |g(u)|\, du$$

$$= |f(y)| \cdot \|g\|_{L^1}$$

Inoltre $\varphi(y) = \int_{\mathbb{R}} h(x,y)\,dx$ è integrabile

$$\int_{\mathbb{R}} \varphi(y)\,dy = \int_{\mathbb{R}} |f(y)| \cdot \|g\|_{L^1}\,dy = \|g\|_{L^1} \cdot \|f\|_{L^1}$$

Per Tonelli la funzione $\underline{h(x,y)}$ è integrabile su $\mathbb{R}^2$

quindi posso usare Fubini $h(x,\cdot)$ è integrabile su $\mathbb{R}$

$|f(\cdot)| \cdot |g(x-\cdot)|$ è integrabile $\int_{\mathbb{R}} f(y)\, g(x-y)\,dy$ esiste

$$\underline{f * g(x)} = \int_{\mathbb{R}} f(y)\, g(x-y)\,dy$$

$f * g \in L^1(\mathbb{R})$

Teorema

f, g trasformabili, allora $f * g$ è trasformabile e $\mathcal{L}\{f*g\}(s) = \mathcal{L}\{f\}(s) \cdot \mathcal{L}\{g\}(s)$

per $Re(s) > \max\{\lambda_f, \lambda_g\}$.

$$e^{-st} = e^{-s\tau} \cdot e^{-s(t-\tau)}$$

Sia $Re(s) > \lambda_{f_1}, > \lambda_g$;

Dim $\mathcal{L}\{f*g\}(t) = \int_{\mathbb{R}} e^{-st} \left(\int_{\mathbb{R}} f(\tau)\, g(t-\tau)\, d\tau \right) dt =$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \underbrace{e^{-s\tau} f(\tau)}_{f_1(\tau)} \cdot \underbrace{e^{-s(t-\tau)} g(t-\tau)}_{g_1(t-\tau)}\, d\tau \right) dt = *$$

$f_1, g_1 \in L^1(\mathbb{R}) \Rightarrow f_1 * g_1$ è integrabile e si può scambiare l'ordine di integrazione

$$* = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-s\tau} f(\tau) \cdot e^{-s(t-\tau)} g(t-\tau)\, dt \right) d\tau =$$

$$= \int_{\mathbb{R}} e^{-s\tau} f(\tau) \cdot \left(\boxed{\int_{-\infty}^{+\infty} e^{-s(t-\tau)} g(t-\tau)\, dt} \right) d\tau = \circledast$$

$t - \tau = u$

$$\int_{-\infty}^{+\infty} e^{-su} g(u)\, du = \underline{\mathcal{L}\{g\}(s)}$$

$$\circledast = \mathcal{L}\{g\}(s) \cdot \underbrace{\int_{\mathbb{R}} e^{-s\tau} f(\tau)\, d\tau}_{= \mathcal{L}\{f\}(s)} = \mathcal{L}\{g\}(s) \cdot \mathcal{L}\{f\}(s).$$

Segnali $f(t)=0$ se $t<0$ $\qquad g(t)=0$ se $t<0$

$$f*g(t)=\int_{-\infty}^{+\infty} f(\tau)\underbrace{u(\tau)}_{=0 \text{ se } \tau<0}\cdot g(t-\tau)\underbrace{u(t-\tau)}_{=0 \text{ se } t-\tau<0}\,d\tau=\int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$t<\tau$

Esempio $f(t)$

$$\varphi(t)=\int_0^t f(\xi)\,d\xi \qquad \varphi(0)=0$$

$$(f*u)(t)=\int_0^t f(\tau)\,1\,d\tau=\varphi(t)$$

$$\mathcal{L}\left\{\int_0^t f(\xi)\,d\xi\right\}(s)=F(s)\cdot\frac{1}{s}$$

$$\mathcal{L}\left\{ \int_0^t f(\xi)\,d\xi \right\}(s) = \frac{1}{s}\,\mathcal{L}\{f\}(s)$$

$$\varphi(t) = \int_0^t f(\xi)\,d\xi \qquad \varphi'(t) = f(t)$$

$$\varphi(0^+) = 0 \qquad \mathcal{L}\{\varphi'(t)\}(s) = s\,\mathcal{L}\{\varphi\}(s) - \varphi(0^+)$$

$$\mathcal{L}\{f\}(s) = s \cdot \mathcal{L}\{\varphi\}(s)$$

$$\mathcal{L}\{\varphi\}(s) = \frac{1}{s}\,F(s)$$