Teoremi del valore finale / iniziale

f trasformabile $\Rightarrow \lim_{Re(s)\to+\infty} \mathcal{L}\{f\}(s) = 0$ "comportamento a ∞"

$f(t)$ $t \approx 0$ $\leadsto$ $F(s)$ $Re(s) \approx \infty$

$f(t)$ $t \approx \infty$ $\leadsto$ $F(s)$ $s \approx 0$

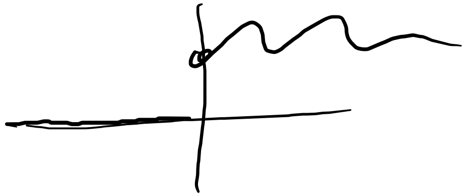
$$\lim_{s\to 0} sF(s) = \lim_{t\to+\infty} f(t) \qquad \lim_{Re(s)\to+\infty} sF(s) = \lim_{t\to 0} f(t)$$

$s \to 0$? $\lambda_f > 0$?

NON VALE IN GENERALE

$f(t) = \operatorname{sen} t \qquad F(s) = \frac{1}{1+s^2} \qquad \lim_{s\to 0} sF(s) = 0 \qquad \lim_{t\to+\infty} \operatorname{sen} t$???

Teorema del valore iniziale



f trasformabile, esiste finito $\lim_{t \to 0^+} f(t) = f(0^+)$

f sia derivabile su $]0,+\infty[$ e f' sia trasformabile

Allora esiste $\lim_{Re(s) \to +\infty} sF(s)$ e si ha $\lim_{Re(s) \to +\infty} sF(s) = f(0^+)$.

Dim

$\mathcal{L}\{f'\}(s) = sF(s) - f(0^+)$ ma $\lim_{Re(s) \to +\infty} \mathcal{L}\{f'\}(s) = 0$

quindi $\lim_{Re(s) \to +\infty} sF(s) = f(0^+)$.

Teorema del valore finale

f trasformabile, f derivabile su $]0,+\infty[$ con f' trasformabile; esiste finito

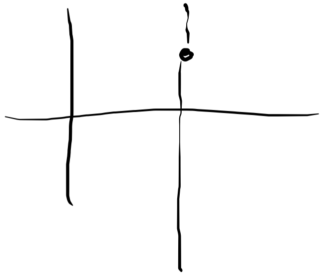
$$\lim_{t \to +\infty} f(t) = f(+\infty).$$

Allora $\lambda_f \leq 0$, $\lambda_{f'} \leq 0$; f' è integrabile su $]0,+\infty[$;

esiste finito $\displaystyle\lim_{\substack{s \to 0 \\ Re(s)>0}} s F(s)$ e si ha $\displaystyle\lim_{\substack{s \to 0 \\ Re(s)>0}} s F(s) = f(+\infty).$

Dim. Proviamo che $\lambda_f \leq 0$, cioè $F(s)$ è definita per ogni s $Re(s)>0$.

Sia $s \in \mathbb{R}$ $s > 0$;



$$\lim_{t \to +\infty} f(t) = f(+\infty)$$

$|f(t) - f(+\infty)| < 1$ se $t > t_0$

$$|f(t)| < \underbrace{1 + |f(+\infty)|}_{K \in \mathbb{R}}$$

$$\left| \int_0^{+\infty} e^{-st} f(t)\, dt \right| \le \underbrace{\int_0^{t_0} |e^{-st} f(t)|\, dt}_{\in \mathbb{R}} + \int_{t_0}^{+\infty} e^{-st} |f(t)|\, dt$$

$t > t_0$

$$\le \int_{t_0}^{+\infty} K e^{-st} < +\infty$$

$s > 0$

quindi F è definita in s

quindi $\underline{\lambda_f \le 0}$

Sia $s \in \mathbb{C}$ $\overline{Re(s) > 0}$ e $Re(s) > \lambda_{f'}$;

$$\boxed{\mathcal{L}\{f'\}(s)}_{\in \mathbb{C}} = \int_0^{+\infty} e^{-st} f'(t)\, dt = \underbrace{\left[e^{-st} f(t) \right]_0^{+\infty}} + \underbrace{s \int_0^{+\infty} e^{-st} f(t)\, dt}_{s\,\mathcal{L}\{f\}(s)}$$

$$\lim_{t \to +\infty} e^{-st} f(t) = 0$$

tende a $f(+\infty)$

$Re(s) > 0$

$$\lim_{t \to 0^+} e^{-st} f(t)$$

?

esiste!

$\in \mathbb{C}$

$$\lim_{t \to 0^+} e^{-st} f(t) = \ell = f(0^+)$$

$$\lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} e^{st} \left(e^{-st} f(t) \right)$$

$$\mathcal{L}\{f'\}(s) = s\,\mathcal{L}\{f\}(s) - \boxed{f(0^+)} \quad \text{se } \operatorname{Re}(s) > 0, \; > \lambda_{f'}$$

Sia s con $\operatorname{Re}(s) > 0$ rifaccio i conti

$$\int_0^{+\infty} e^{-st} f'(t)\,dt = \underbrace{\lim_{t\to+\infty} f(t)\,e^{-st}}_{=\,0} - \underbrace{\lim_{t\to 0^+} f(t)\,e^{-st}}_{=\,f(0^+)} + s\,\underbrace{\mathcal{L}\{f\}(s)}_{\text{esiste}}$$

$\Rightarrow$ esiste! quindi $\lambda_{f'} \leq 0$.

$\underline{f' \text{ integrabile}}$

$$\int_0^{+\infty} f'(t)\,dt = \underbrace{\lim_{t\to+\infty} f(t)}_{\text{esiste}} - \underbrace{\lim_{t\to 0^+} f(t)}_{\text{esiste}} \in \mathbb{C}$$

Infine calcoliamo

$$\lim_{\substack{s\to 0\\ Re(s)>0}} \underbrace{\mathcal{L}\{f'(t)\}(s)}_{sF(s)-f(0^+)} = \lim_{\substack{s\to 0\\ Re(s)>0}} \int_0^{+\infty} e^{-st} f'(t)\,dt \underset{\text{Lebesgue}}{=} \int_0^{+\infty} \lim_{\substack{s\to 0\\ Re(s)>0}} \overbrace{e^{-st}}^{1} f'(t)\,dt = *$$

$$* = \int_0^{+\infty} f'(t)\,dt = f(+\infty) - f(0^+)$$

Quindi $\lim_{\substack{s\to 0\\ Re(s)>0}} \left(sF(s) - f(0^+)\right) = f(+\infty) - f(0^+)$

$\Rightarrow$ Tesi

$$\lim_{\substack{s \to 0 \\ \mathrm{Re}(s) > 0}} \int_0^{+\infty} e^{-st} f'(t)\, dt \overset{?}{=} \int_0^{+\infty} \lim_{\substack{s \to 0 \\ \mathrm{Re}(s) \to 0}} e^{-st} f'(t)\, dt$$

$$|e^{-st} f'(t)| \leq |f'(t)| \text{ integrabile}$$

$$\mathrm{Re}(s) > 0 \Rightarrow |e^{-st}| \leq 1$$

Seno cardinale $\quad \mathrm{sinc}(t) = f(\pi t) \qquad f(t) = \begin{cases} \frac{\mathrm{sen}\, t}{t} & \text{se } t > 0 \\ 1 & \text{se } t = 0 \\ 0 & \text{se } t < 0 \end{cases}$

$\lambda_f = 0$

$f(t)$ non è integrabile su $]0, +\infty[$ però $\forall s \; Re(s) > 0 \quad e^{-st} f(t)$ è integrabile

(f è limitata)

$\mathcal{L}\{f\}(s) = \underline{F(s)} = ?$

$$F'(s) = -\mathcal{L}\{t \cdot f(t)\}(s) = -\mathcal{L}\{\mathrm{sen}\, t \, u(t)\}(s) = -\frac{1}{1+s^2}$$

$s = x \in \mathbb{R}$

$$F(x) - F(0) = \int_0^x F'(\xi)\, d\xi = \int_0^x -\frac{1}{1+\xi^2}\, d\xi = -\mathrm{arctg}(x)$$

$F(x) = F(0) - \mathrm{arctg}\, x \qquad F(s) = F(0) - \mathrm{arctg}(s) \qquad s \in \mathbb{C} \quad Re(s) > 0$

??

$\lim_{t \to +\infty} \mathrm{sinc}(t) = 0$

$$0 = \lim_{Re(s) \to +\infty} F(s) = F(0) - \frac{\pi}{2}$$

$\boxed{F(0) = \frac{\pi}{2}}$

$$\mathcal{L}\left\{ \frac{\operatorname{sen} t}{t} u(t) \right\} = \frac{\pi}{2} - \operatorname{arctg}(s)$$

$$\mathcal{L}\left\{ \operatorname{sinc}(t) \right\}(s) = \frac{1}{\pi} F\left(\frac{s}{\pi}\right) = \frac{1}{2} - \frac{1}{\pi} \operatorname{arctg}\left(\frac{s}{\pi}\right)$$

$$\overset{\shortparallel}{f(\pi s)}$$

Seno integrale $\quad Si(t) = \int_0^t \frac{\operatorname{sen} \tau}{\tau} d\tau$

$$\mathcal{L}\left\{ Si(t) \right\}(s) = \frac{1}{s} \mathcal{L}\left\{ f(t) \right\}(s) = \frac{1}{s}\left(\frac{\pi}{2} - \operatorname{arctg}(s) \right)$$

$$\boxed{\int_0^{+\infty} \frac{\operatorname{sen} t}{t} dt} = \lim_{t \to +\infty} Si(t) = \lim_{\substack{s \to 0 \\ Re(s) > 0}} s F(s) = \lim_{\substack{s \to 0 \\ Re(s) > 0}} \frac{\pi}{2} - \operatorname{arctg}(s) = \frac{\pi}{2}$$

Antitrasformata

$$\begin{tikzcd} f(t) \arrow[rr, "\mathcal{L}"'] & & F(s) \arrow[ll, bend left=60, "\mathcal{L}^{-1}"] \end{tikzcd}$$

Problemi per definire $\mathcal{L}^{-1}$

$\mathcal{L} : \{\text{segnali trasformabili}\} \to \{\text{funzioni analitiche definite su semipiani}\}$

??

$\mathcal{L}^{-1} : \{F \text{ trasformata di qualche segnale}\} \to \{\text{segnali}\}$

?

$\mathcal{L}$ è iniettivo? No! se $f(t) = g(t)$ q.o. $\Rightarrow \mathcal{L}\{f\} = \mathcal{L}\{g\}$

Possiamo limitarci a considerare funzioni trasformabili continue a tratti

Teorema

Siano f e g continue (segnali) su $\mathbb{R}$, con al più un insieme discreto di salti,

sia $\mathcal{L}\{f\} = \mathcal{L}\{g\}$. Allora $f = g$.

È allora possibile definire la trasformata inversa

$\mathcal{L}^{-1}\{F\} = f$ con f continua a tratti se $\mathcal{L}\{f\} = F$.

— o —

$$\mathcal{L}\{\operatorname{sen}(\omega t)\}(s) = \frac{\omega}{s^2+\omega^2}$$

$$F(s) = \frac{2}{s^2+9} \quad \Big\} \quad \frac{2}{3}\,\frac{3}{s^2+9} \rightsquigarrow f(t) = \frac{2}{3}\operatorname{sen}(3t)\,u(t)$$

$$F(s) = \frac{3s+1}{s^2+4} = 3\,\frac{s}{s^2+4} + \frac{1}{2}\,\frac{2}{s^2+4} \rightsquigarrow 3\cos(2t) + \frac{1}{2}\sin(2t)$$

$$F(s) = |s| \rightsquigarrow f(t) = ?$$ non esiste!

$$F(s) = e^{s} \rightsquigarrow$$ NON esiste

$$F(s) = e^{-s} \operatorname{arctg} s \quad ?$$

Formula per l'antitrasformata?

Formula di Bromwich-Mellin ; Riemann-Fourier

$f : \mathbb{R} \to \mathbb{C}$ segnale continuo con al più un insieme discreto di salti, sia

$F(s) = \mathcal{L}\{f(t)\}(s)$, allora $\forall t \in \mathbb{R}$

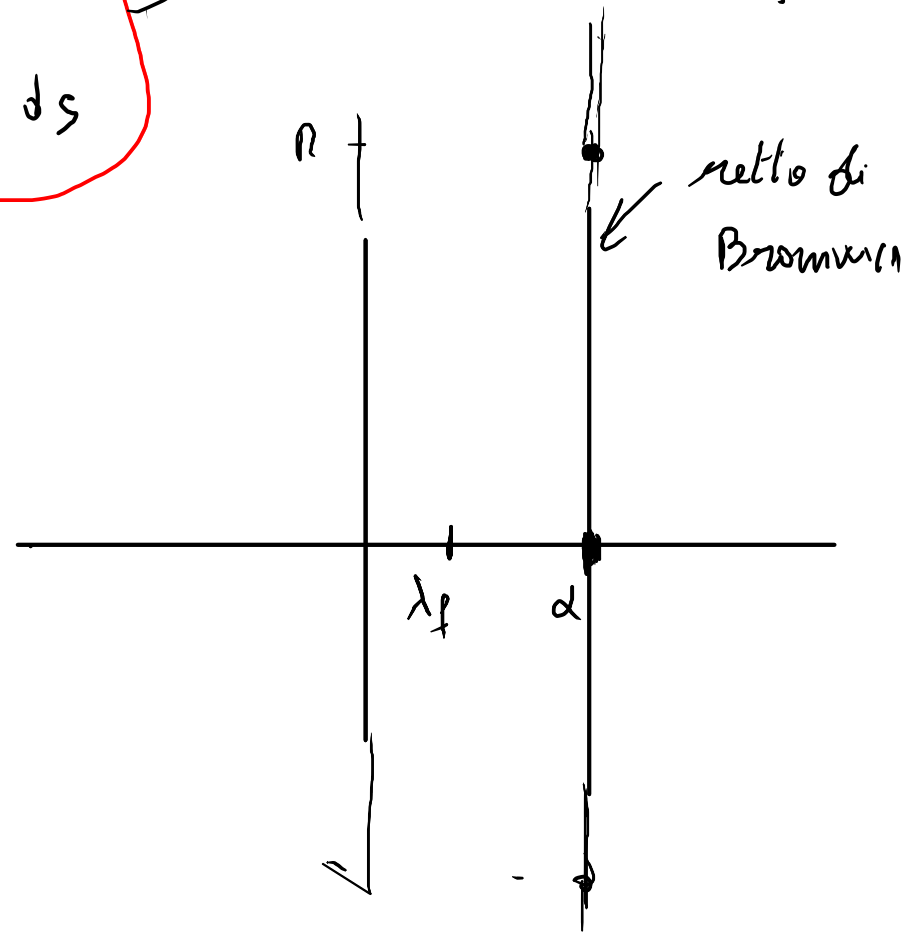
con $\alpha \in \mathbb{R}$ $\alpha > \lambda_f$

$$\frac{1}{2}\left(f(t^+) + f(t^-)\right) = \boxed{\frac{1}{2\pi i} \lim_{R \to +\infty} \int_{\alpha - iR}^{\alpha + iR} e^{st} F(s)\, ds}$$

$= f(t)$ se f è continua in t

$$f(t^+) = \lim_{\tau \to t^+} f(\tau)$$

$$f(t^-) = \lim_{\tau \to t^-} f(\tau)$$



La formula si ottiene grazie alla formula per l'antitrasformata di Fourier

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}\{g(t)\}(y)\, e^{ity}\, dy$$

$$\underbrace{\mathcal{L}\{f(t)\}(x+iy)} = \underbrace{\mathcal{F}\{ \underbrace{e^{-xt} f(t) u(t)}_{= g(t)} \}(y)}$$

$x = \alpha$

$$e^{-\alpha t} f(t) u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{\mathcal{F}\{e^{-\alpha t} f(t) u(t)\}(y)}_{= \mathcal{L}\{f(t)\}(\alpha+iy)}\, e^{ity}\, dy =$$

$$e^{-\alpha t} f(t) u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\alpha+iy)\, e^{ity}\, dy$$

$s = \alpha + iy \qquad ds = i\, dy$

$$f(t) u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\alpha+iy)\, e^{(\alpha+iy)t}\, dy = = \frac{1}{2\pi i} \lim_{R \to +\infty} \int_{\alpha - iR}^{\alpha + iR} F(s)\, e^{st}\, ds$$

Scorciatoia $\quad F(s) = \frac{1}{s+9} e^{-5s}$

$$\frac{1}{s} \rightsquigarrow u(t)$$

$$\frac{1}{s+9} \rightsquigarrow e^{-9t} u(t)$$

$$e^{-5s} \frac{1}{s+9} \rightsquigarrow \underbrace{e^{-9(t-5)} u(t-5)}$$

$$\frac{1}{s} \qquad u(t)$$

$$e^{-5s} \frac{1}{s} \rightsquigarrow u(t-5)$$

$$\underbrace{e^{-5(s+9)} \frac{1}{s+9}} \rightsquigarrow e^{-9t} u(t-5)$$

$$e^{-5s-45} \frac{1}{s+9} = e^{-45} \cdot \left(e^{-5s} \frac{1}{s+9} \right)$$

$$\rightsquigarrow e^{45-9t} u(t-5)$$

$$F(s) = \frac{1}{s^3+12s^2+36s} = \frac{a}{s} + \frac{b}{s+6} + \frac{c}{(s+6)^2} = *$$

$$s(s^2+12s+36) = s(s+6)^2$$

$$a = \text{Res}(F, 0) = \frac{1}{36}$$

$$c = \lim_{s\to-6} F(s)(s+6)^2 = -\frac{1}{6}$$

$$\frac{1}{s}$$

$$* = \frac{1/36}{s} + \frac{-1/36}{s+6} + \frac{-1/6}{(s+6)^2}$$

$$b = \text{Res}(F,-6) = \lim_{s\to-6} D\left[F(s)(s+6)^2\right] = -\frac{1}{36}$$

$$\frac{1}{s}$$

$$\frac{1}{s^2} \rightsquigarrow t\,u(t)$$

$\mathcal{L}^{-1}$

$$\frac{1}{36}u(t) - \frac{1}{36}e^{-6t}u(t) - \frac{1}{6}e^{-6t}t\,u(t)$$

Applicazioni

$$\begin{cases} y'' + c_1 y' + c_0 y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

$\updownarrow$ equivalenti

$\mathcal{L} \rightsquigarrow$

$Y = \mathcal{L}\{y\}$

$$\underbrace{\mathcal{L}\{y''\}}_{= s^2 Y - s\,y(0^+) - y'(0^+)} + c_1 \underbrace{\mathcal{L}\{y'\}}_{= sY - y(0^+)} + c_0 \underbrace{\mathcal{L}\{y\}}_{= Y} = \underbrace{\mathcal{L}\{0\}}_{= 0}$$

$y(0^+) = 0$, $y'(0^+) = 1$

$$\begin{cases} y'' + c_1 y' + c_0 y = \delta_0 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}$$

? $\mathcal{L} \rightsquigarrow$

$$s^2 Y + c_1 s Y + c_0 Y = 1 \leftarrow \mathcal{L}\{\delta_0\}$$

$$Y = \frac{1}{s^2 + c_1 s + c_0}$$

risposta impulsiva

$$y^{(n)} + c_{n-1} y^{(n-1)} + c_{n-2} y^{(n-2)} + \dots + c_1 y' + c_0 y = L y$$

$$\begin{cases} L y = 0 \\ y(0) = 0, \dots, y^{(n-2)}(0) = 0 \\ y^{(n-1)}(0) = 1 \end{cases} \quad \rightsquigarrow \quad \begin{cases} L y = \delta_0 \\ y(0) = \dots = y^{(n-1)}(0) = 0 \end{cases}$$

$$Y = \frac{1}{s^n + c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \dots + c_0}$$

$f(t)$