$$Ly = y^{(n)} + c_{n-1}\, y^{(n-1)} + \dots + c_1 y' + c_0 y \qquad L : C^n(]0,+\infty[, \mathbb{R}) \to C^0(]0,+\infty[, \mathbb{R})$$

$$\begin{cases} Ly = 0 \\ y(0) = y_0 \\ y'(0) = y_1 \\ y''(0) = y_2 \\ \vdots \\ y^{(n-1)}(0) = y_{n-1} \end{cases}$$

$n = 3$ $\mathcal{L} \rightsquigarrow$

$$\underbrace{\left[s^3 Y - s^2 y(0^+) - s y'(0^+) - y''(0^+) \right]}_{\mathcal{L}\{y'''\}(s)} +$$

$$+ c_2 \left[s^2 Y - s\, y(0^+) - y'(0^+) \right] + c_1 \left[sY - y(0^+) \right] + c_0 Y = 0$$

$$Y \underbrace{\left(s^3 + c_2 s^2 + c_1 s + c_0 \right)}_{1/R(s)} = \underbrace{\left(s^2 y_0 + s y_1 + y_2 \right) + c_2 \left(s y_0 + y_1 \right) + c_1 y_0}_{P(s)}$$

$P(s)$ ← non è la trasformata di una funzione

$$Y = R(s) \cdot P(s)$$

$$\boxed{y(t) = \mathcal{L}^{-1}\{ R(s) \cdot P(s) \}(t)} \overset{?}{=} \underbrace{\mathcal{L}^{-1}\{ R(s) \}(t)}_{r(t)} * \cancel{\mathcal{L}^{-1}\{ P(s) \}(t)} \quad !!$$

$$\begin{cases} Ly = f(t) \\ y(0) = y'(0) = y''(0) = \cdots = y^{(n-1)}(0) = 0 \end{cases}$$

$$\boxed{Y(s) = R(s) \cdot F(s)}$$

$$\boxed{y(t) = r * f}$$

$$\begin{cases} Ly = f(t) \\ y(0) = y_0 \\ y'(0) = y_1 \\ \vdots \\ y^{(n-1)}(0) = y_{n-1} \end{cases}$$

$$Y(s) = R(s) \cdot F(s) + R(s) \cdot P(s)$$

$$\boxed{y(t) = r * f(t) + \mathcal{L}^{-1}\{R(s) \cdot P(s)\}(t)}$$

$$\begin{cases} y'' + 3y' + 2y = e^t \\ y(0) = 0 \quad y'(0) = 1 \end{cases}$$

$$\mathcal{L} \rightsquigarrow \left[s^2 Y - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=1} \right] + 3\left[sY - \underbrace{y(0)}_{=0} \right] + 2Y = \frac{1}{s-1}$$

$$\mathcal{L}\{e^t\}(s) = \frac{1}{s-1}$$

$$e^t u(t)$$

$$e^{\omega t} f(t) \rightsquigarrow F(s-\omega)$$

$$Y(s^2+3s+2) = \frac{1}{s-1} + 1 = \frac{1+s-1}{s-1} = \frac{s}{s-1}$$

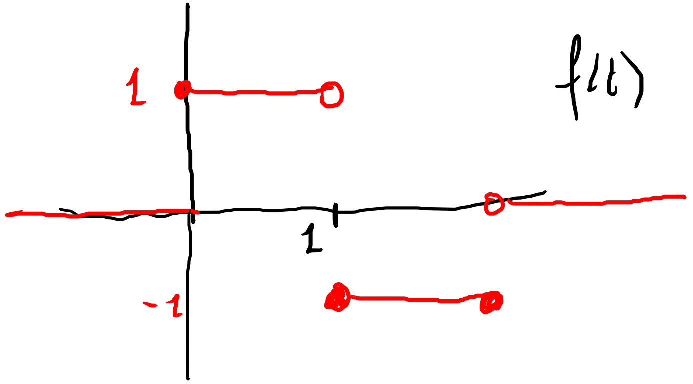
$$\frac{1}{s-1} \rightsquigarrow e^t$$

$$Y = \frac{s}{s-1} \cdot \frac{1}{\underbrace{s^2+3s+2}_{=(s+1)(s+2)}} = \frac{\cancel{A}\ 1/6}{s-1} + \frac{\cancel{B}\ 1/2}{s+1} + \frac{\cancel{C}\ -2/3}{s+2}$$

$$\frac{1}{2\cdot 3} \qquad \frac{-1}{-2\cdot 1} \qquad \frac{-2}{-3\cdot(-1)} = -\frac{2}{3}$$

$$y(t) = \left(\frac{1}{6} e^t + \frac{1}{2} e^{-t} - \frac{2}{3} e^{-2t} \right) u(t)$$

$$\begin{cases} y'' + y = f(t) \\ y(0) = y'(0) = 0 \end{cases}$$



$$f(t) = \begin{cases} 1 & \text{se } t \in [0,1[\\ -1 & \text{se } t \in [1,2] \\ 0 & \dots \end{cases}$$

Soluzione in senso distribuzionale

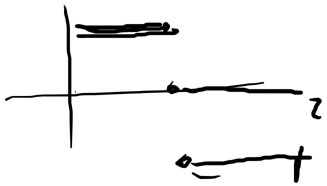
$\mathcal{L} \rightsquigarrow \quad s^2 Y + Y = F(s) \qquad Y(s) = \frac{1}{1+s^2} \cdot F(s) \qquad y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{1+s^2}\right\}(t)}_{\text{sen } t \cdot u(t)} * f(t)$

$$y(t) = \text{sen } t * f(t)$$

$$= \int_0^t f(\tau) \,\text{sen}(t-\tau)\, d\tau = \dots$$

oppure $f(t) = \underbrace{u(t) - u(t-1)}_{} + $

$\hookrightarrow - u(t-1) + u(t-2)$



$$f(t) = u(t) - 2u(t-1) + u(t-2)$$

$$F(s) = \frac{1}{s} - 2e^{-s}\frac{1}{s} + e^{-2s}\frac{1}{s}$$

$$Y(s) = \frac{1}{s^2+1} \cdot \frac{\left(1-2e^{-s}+e^{-2s}\right)}{s} = \underbrace{\frac{1}{(s^2+1)\,s}}_{\left[\frac{1}{s} - \frac{s}{s^2+1}\right]} \cdot \left(1-2e^{-s}+e^{-2s}\right)$$

$$\left[\frac{1}{s} - \frac{s}{s^2+1}\right] \rightsquigarrow u(t) - \cos(t)\,u(t)$$

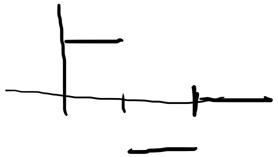
$$y(t) = \left[u(t) - \cos(t)\,u(t)\right] - 2\left[u(t-1) - \cos(t-1)\,u(t-1)\right] + \left[u(t-2) - \cos(t-2)\,u(t-2)\right]$$

$$\int_0^t f(\tau)\,\text{sen}(t-\tau)\,d\tau =$$

se $t<1$: $= \int_0^t \text{sen}(t-\tau)\,d\tau = \left[\cos(t-\tau)\right]_0^t = 1-\cos(t)$

se $1<t<2$: $= \int_0^1 \text{sen}(t-\tau)\,d\tau - \int_1^t \text{sen}(t-\tau)\,d\tau = \left[\cos(t-\tau)\right]_0^1 - \left[\cos(t-\tau)\right]_1^t$

$= \cos(t-1) - \cos(t) - 1 + \cos(t-1) = 2\cos(t-1) - \cos(t) - 1$



se $t>1$: $\int_0^1 \text{sen}(t-\tau)\,d\tau - \int_1^2 \text{sen}(t-\tau)\,d\tau = \ldots$

Sistemi lineari

$$\begin{cases} x' = Ax + f \\ x(0) = x_0 \end{cases}$$

$x : \mathbb{R} \to \mathbb{R}^N$ $\quad$ A matrice $N \times N$ $\quad$ $f = (f_1, \dots, f_N)$

$\begin{pmatrix} x \\ y \end{pmatrix}$ $N = 2$

$$\begin{cases} x' = 2y \\ y' = 4x - 2y \\ x(0) = 1, \quad y(0) = 0 \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 2 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

L $\mathcal{L}$

$$\begin{cases} sX - \overset{=1}{x(0^+)} = 2Y \\ sY - \underset{=0}{y(0^+)} = 4X - 2Y \end{cases} \qquad \begin{cases} sX - 1 = 2Y \\ sY = 4X - 2Y \end{cases} \leadsto Y = \frac{1}{2}(sX - 1)$$

$$\leadsto \frac{1}{2} s (sX - 1) = 4X - (sX - 1)$$

$$\frac{1}{2} s^2 X - \frac{1}{2} s = 4X - sX + 1$$

$$s^2 X - s - 8X + 2sX = 2$$

$$X(s^2 - 8 + 2s) = 2 + s$$

$$\boxed{X = \frac{s+2}{s^2 + 2s - 8}}$$

$$X = \frac{s+2}{\underset{(s+4)(s-2)}{\underbrace{s^2+2s-8}}} = \frac{s+2}{(s+4)(s-2)}$$

$$\operatorname{Res}(-4) = \frac{-2}{-6}$$

$$\operatorname{Res}(2) = \frac{4}{6}$$

$$X = \frac{1/3}{s+4} + \frac{2/3}{s-2}$$

$$Y = \frac{-2/3}{s+4} + \frac{2/3}{s-2}$$

$$Y = \frac{1}{2}\left(s \cdot \frac{s+2}{(s+4)(s-2)} - 1 \right)$$

$$= \frac{1}{2}\left(\frac{\cancel{s^2} + \cancel{2s} - \cancel{s^2} - \cancel{2s} + 8}{(s+4)(s-2)} \right) = \frac{4}{(s+4)(s-2)}$$

$$\frac{4}{-6}$$

$$x(t) = \frac{1}{3} e^{-4t} u(t) + \frac{2}{3} e^{2t} u(t)$$

$$y(t) = -\frac{2}{3} e^{-4t} u(t) + \frac{2}{3} e^{2t} u(t)$$

Equazioni integro-differenziali

$$y(t) = \cos(t) + \underbrace{\int_0^t (t-\tau)\, y(\tau)\, d\tau}_{y * t\,u(t)} \quad \overset{\mathcal{L}}{\rightsquigarrow} \quad Y(s) = \underbrace{\mathcal{L}\{\cos t\}(s)}_{\frac{s}{1+s^2}} + \underbrace{\mathcal{L}\{y * t\,u(t)\}(s)}_{Y \cdot \underbrace{\mathcal{L}\{t\,u(t)\}}_{=\frac{1}{s^2}}}$$

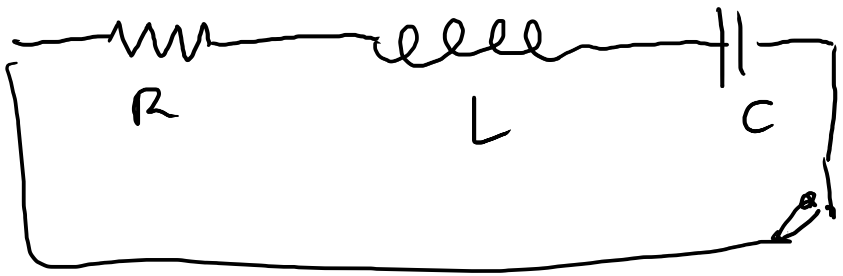
$$Y = \frac{s}{1+s^2} + Y\frac{1}{s^2} \qquad Y\underbrace{\left(1-\frac{1}{s^2}\right)}_{=\frac{s^2-1}{s^2}} = \frac{s}{1+s^2}$$

$$Y = \frac{s}{1+s^2} \cdot \frac{s^2}{s^2-1} = \frac{s^3}{\underbrace{(1+s^2)}_{(s+i)(s-i)}(s-1)(s+1)}$$

$$Y = \frac{1}{4}\left(\frac{1}{s-1} + \frac{1}{s+1} + \frac{1}{s-i} + \frac{1}{s+i}\right)$$

$$y(t) = \frac{1}{4}\left(e^{t} + e^{-t} + \underline{e^{it} + e^{-it}}\right) = \frac{1}{2}\left(\cosh(t) + \cos(t)\right)$$

Circuiti elettrici RLC



legge di Kirchhof

$$L \frac{di}{dt} + Ri + \frac{1}{C} q = \underset{\sim}{v}$$

$$q(t) = q_0 + \int_0^t i(\tau)\, d\tau$$

i corrente

$$L i' + Ri + \frac{1}{C} q_0 + \frac{1}{C} \int_0^t i(\tau)\, d\tau \quad \underset{\mathcal{L}}{\rightsquigarrow} \quad sLI + RI + \frac{1}{Cs} I + \frac{q_0}{C} \frac{1}{s} = V(s)$$

$i(0^+) = 0$

$$I \left(sL + R + \frac{1}{Cs} \right) = V(s) - \frac{q_0}{C} \frac{1}{s}$$

$$I = \frac{V(s) - q_0/sC}{sL + R + \frac{1}{Cs}}$$

$$\frac{1}{sL + R + \frac{1}{Cs}} = \underbrace{T(s)}_{}\ \ Y(s)$$

ammettenza di trasferimento

$$I(s) = \underbrace{T(s)}_{} \cdot \left(V(s) - \frac{q_0}{sC} \right)$$

se $q_0 = 0$

$$i(t) = \mathcal{L}^{-1}\{T(s)\}(t) * v(t)$$

Derivate parziali

$u : A \times \mathbb{R} \to \mathbb{C}$ $\quad u(x,t)$ $\quad u(x,t) = 0 \quad \forall t<0 \quad \forall x \in A$

$A \subseteq \mathbb{R}^N$ $\quad u(x,\cdot)$ segnale

$\forall x$ si può considerare $U(x,s)$ $\quad \lambda_{u(x,\cdot)}$

supponiamo che l'ascissa di convergenza sia uniforme rispetto a x $\quad \lambda_u$

$$U(x,s) = \int_0^{+\infty} e^{-st} u(x,t)\, dt$$

$\boxed{\dfrac{\partial u}{\partial t}}$ $\quad \dfrac{\partial u}{\partial x}$ $\quad \nabla_x u \quad \Delta_x u = \sum_{k=1}^{N} \dfrac{\partial^2 u(x_1,\dots,x_N;t)}{\partial x_k^2}$

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}(x,t)\right\}(x,s) = s\,U(x,s) - \underline{u(x,0^+)}$$

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}(x,t)\right\}(x,s) = \int_0^{+\infty} e^{-st}\,\frac{\partial u}{\partial x}(x,t)\,dt =$$

$$= \frac{\partial}{\partial x}\int_0^{+\infty} e^{-st}\,u(x,t)\,dt = \frac{\partial}{\partial x}U(x,t)$$

Lebesgue

$$s_0 = \lambda_{\frac{\partial u}{\partial x}}$$

$$\left|e^{-st}\,\frac{\partial u}{\partial x}(x,t)\right| \le \underbrace{\left|e^{-s_0 t}\,\frac{\partial u}{\partial x}(x,t)\right|}_{\text{intégrable}}$$

Eq. del calore su un filo infinito

$u(x,t)$ = calore al tempo t nel punto x

$u(x,0)=0 \quad \forall x \in [0,+\infty[$ condizione iniziale

$u(0,t) = \varphi(t) \qquad \lim\limits_{x\to+\infty} u(x,t) = 0 \quad \forall t$ [condizioni al contorno]

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(x,0)=0 \\ u(0,t) = \varphi(t), \quad \lim\limits_{x\to+\infty} u(x,t)=0 \end{cases} \quad \overset{\mathcal{L}}{\rightsquigarrow} \quad \begin{cases} \overset{s\cdot z}{sU(x,s)} - \underbrace{u(x,0^+)}_{=0} = \overset{z''}{\dfrac{\partial^2}{\partial x^2} U(x,s)} \\ \underset{z(0)\,=}{U(0,s)} = \Phi(s) \qquad \mathcal{L}\{\lim\limits_{x\to+\infty} u(x,t)\}(s) = 0 \end{cases}$$

$$\lim_{x\to+\infty} U''(x,s) = 0$$

Fissiamo $s \in \mathbb{R}^+$ $\quad z(x) = U(x, s)$

$$\begin{cases} s z = z'' \\ z(0) = \phi(s) \\ \lim_{x \to +\infty} z(x) = 0 \end{cases}$$

$z'' = s z \qquad s \in \mathbb{R}$ $\qquad (L y = \lambda y)$

$\lambda^2 - s = 0 \qquad \lambda = \pm \sqrt{s}$

$z(x) = \lambda_1 e^{-\sqrt{s} x} + \lambda_2 e^{\sqrt{s} x}$

$\lim_{x \to +\infty} z(x) = 0 \implies \lambda_2 = 0$

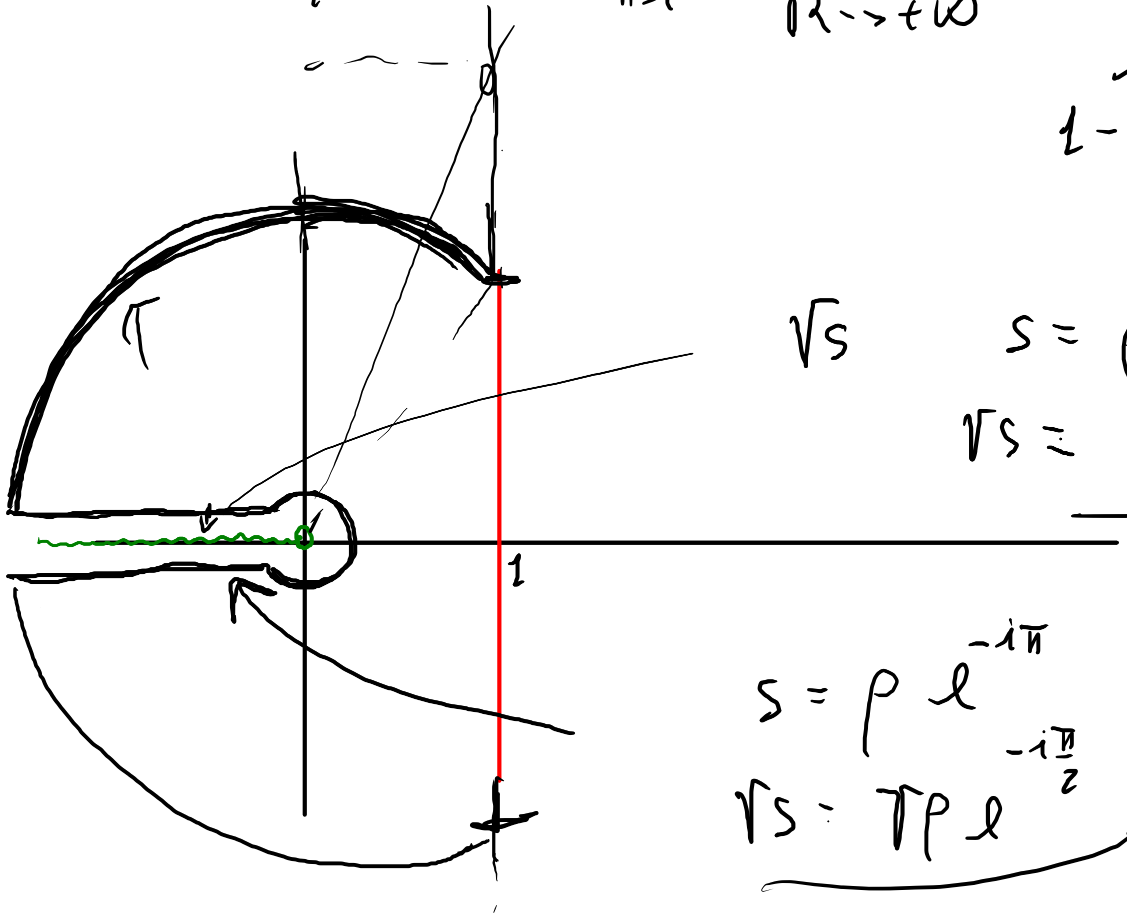
$\phi(s) = z(0) = \lambda_1$

$z(x) = \phi(s) e^{-\sqrt{s} x}$

$U(x, s) = \phi(s) e^{-\sqrt{s} x}$

$$\boxed{u(x,t) = \mathcal{L}^{-1}\{ e^{-\sqrt{s} x} \}(t) * \varphi(t)}$$

$$\mathcal{L}^{-1}\{e^{-\sqrt{s}\,x}\}(t) = \frac{1}{2\pi i}\lim_{R\to+\infty}\int_{1-iR}^{1+iR} e^{st}\,e^{-\sqrt{s}\,x}\,ds$$



$\sqrt{s}$

$s = \rho e^{i\pi}$

$\sqrt{s} = \sqrt{\rho}\, e^{i\frac{\pi}{2}}$

$s = \rho e^{-i\pi}$

$\sqrt{s} = \sqrt{\rho}\, e^{-i\frac{\pi}{2}}$

$s = 1 + iy$

$e^{st} = e^{t} + e^{iyt}$

$s = x + iy \quad x \leq 1$

$e^{st} = e^{xt + iyt}$

$$\mathcal{L}^{-1}\left\{ e^{-\sqrt{s}\,x} \right\} = \frac{x}{2\sqrt{\pi}\, t^{3/2}}\, e^{-\frac{x^2}{4t}}$$