1st lesson (Adv. Analysis)

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monday 11 - 13 (3B)

thur. 10,45 - 12,30 (room C C9 building)

didactical material : <u>Moodle</u>

CONTENT OF THE COURSE

i) differentiation theory (Hewitt-Stromberg)

ii) distribution theory (Hörmander's style)

iii) Sobolev spaces (Brezis)

8-9 chapt.

1. continuous functions which are not differentiable at any point (continuous nowhere diff. funct.)

- $f: I \longrightarrow \mathbb{R}$ $x_0 \in I$

I open interval

i) f continuous at x_0 means $\lim_{x \to x_0} f(x) = f(x_0)$

ii) f differentiable at x_0 " $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \in \mathbb{R}$

ii) $\Rightarrow$ i) and i) $\not\Rightarrow$ ii)

e.g. $x \longrightarrow |x|$ is cont. at 0, not diff.

Problem: find a cont. funct (in $\mathbb{R}$) without any point in which it is diff.

Answer: (Weierstrass 1875)

Th. let $$f(x) = \sum_{n=0}^{+\infty} b^n \cos(a^n \pi x)$$

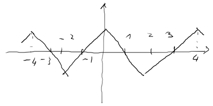
with $b \cdot a > 1 + \frac{3}{2}\pi$, $b < 1$.

f is cont. and nowhere diff.

(Hardy (1918) it is sufficient that $ab > 1$)

Ex. (McCarthy Am. Math. Monthly 1953)

$$g(x) = \begin{cases} 1-x & \text{if } x \in [0,2] \\ 1+x & \text{if } x \in [-2,0] \\ \text{periodic of period } 4 \end{cases}$$

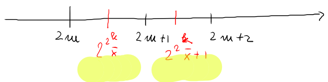


$$f(x) = \sum_{n=1}^{+\infty} 2^{-n} g\left(2^{2^n} x\right)$$

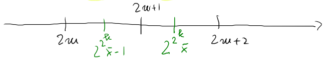
1) f is continuous (f is the uniform limit of continuous funct.)

2) take $\bar{x} \in \mathbb{R}$ and take $k \in \mathbb{N}$

consider $2^{2^k}\bar{x} \in \mathbb{R}$ so that two possibilities



or



Suppose the first take $h = 2^{-2^k}$ (if the second $h = -2^{-2^k}$)

$$g\left(2^{2^n}(\bar{x}+h)\right) - g\left(2^{2^n}\bar{x}\right) = \begin{cases} 0 & n > k \\ 1 & n = k \\ \leq 2^{2^n - 2^k} & n < k \end{cases}$$

$$2^{2^n}\bar{x} + \underbrace{2^{2^n-2^k}}_{\geq 4}$$

$$2^{2^k}\bar{x}+1 \qquad 2^{2^k}\bar{x}$$

$$2^{2^n-2^k} < 1$$

$$f(\bar{x}+h) - f(\bar{x}) = \sum_{n=1}^{+\infty} 2^{-n}\left(g\left(2^{2^n}(\bar{x}+h)\right) - g\left(2^{2^n}\bar{x}\right)\right)$$

$$= \sum_{n=1}^{k} 2^{-n}\left(g\left(2^{2^n}(\bar{x}+h)\right) - g\left(2^{2^n}x\right)\right)$$

$$|f(\bar{x}+h) - f(\bar{x})| \geq 2^{-k} - \sum_{n=1}^{k-1} 2^{-n}\left(2^{2^n-2^k}\right)$$

$$\geq 2^{-k} - \sum_{n=1}^{k-1} 1 \cdot 2^{2^{k-1}-2^k} = 2^{-k} - (k-1)\,2^{-2^{k-1}}$$

$$\frac{|f(\bar{x}+h) - f(\bar{x})|}{h} \geq \underbrace{\left(2^{-k} - (k-1)\,2^{-2^{k-1}}\right) \cdot 2^{2^k}}$$

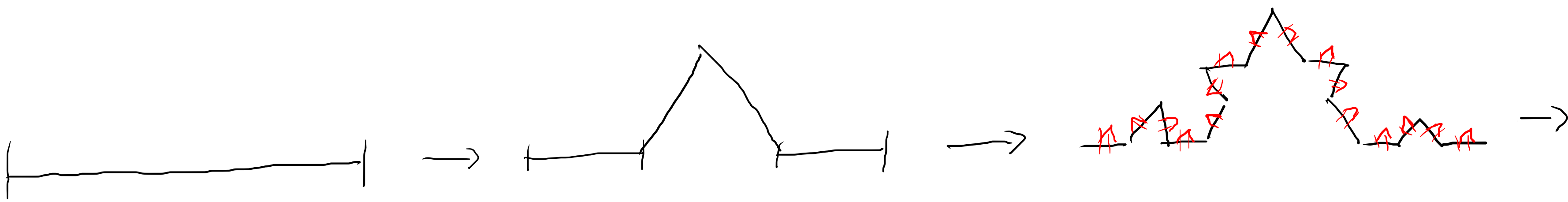
Recall if $h = -2^{-2^k}$ we obtain the same

$$\downarrow k \to +\infty$$

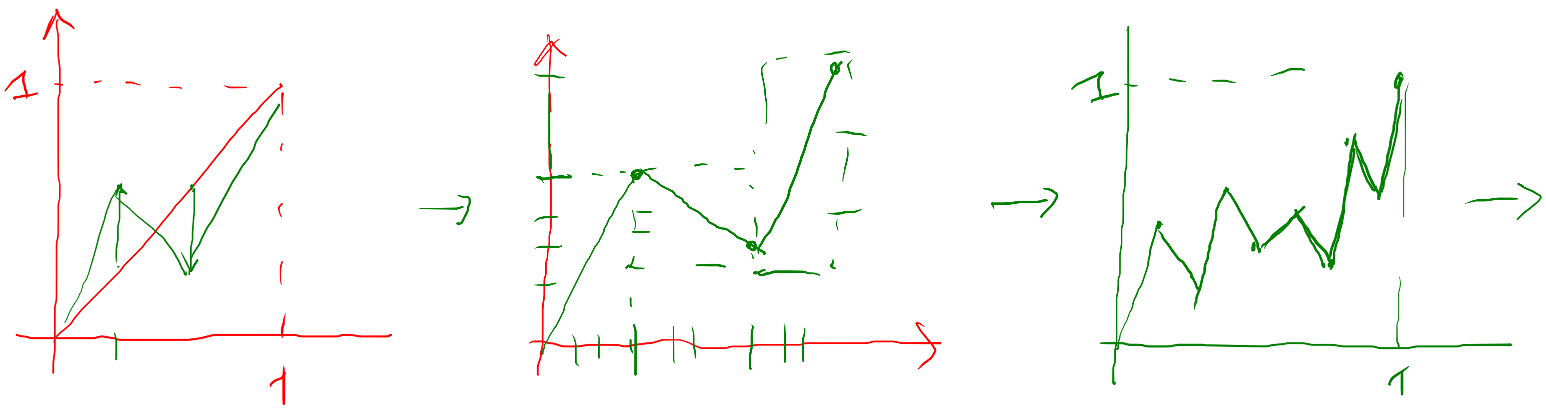
$+\infty$ (exercise)

Rem: in the case of a curve

Von Koch curve



Katsuura



2. How many continuous nowhere diff. funct. are there?

def. let $f: I \to \mathbb{R}$, I open interval, let $x_0 \in I$

define

$$D^+ f(x_0) = \limsup_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$$

$$D_+ f(x_0) = \liminf_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$$

$$D^- f(x_0) = \limsup_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$$

$$D_- f(x_0) = \liminf_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$$

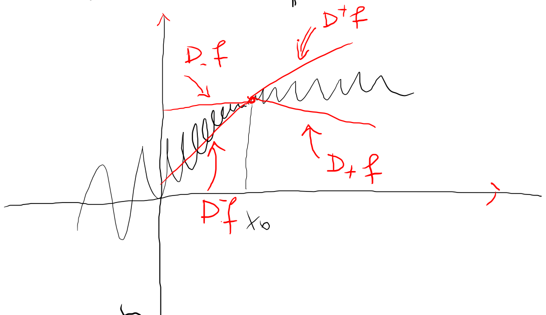
Dini's derivatives

$$\limsup_{x \to x_0^+} g(x) = \inf_{t>0} \left(\sup_{x_0 < x < x_0 + t} g(x) \right)$$

1) $D^+ f(x_0) \geq D_+ f(x_0)$

$D^- f(x_0) \geq D_- f(x_0)$

2) f is diff. $\iff$ $D_- f(x_0) = D^- f(x_0) = D_+ f(x_0) = D^+ f(x_0) \in \mathbb{R}$



Th Denote $\mathcal{C}([0,1])$ the space of cont. functions on $[0,1]$ with the sup-norm.

$$D = \{ f \in \mathcal{C}([0,1]) : \exists x \in [0,1[: D_+ f(x), D^+ f(x) \in \mathbb{R} \}$$

Then D is contained in the countable union of closed set with empty interior proof.

Let $C_n = \{ f \in \mathcal{C}([0,1]) : \exists x \in [0, 1-\frac{1}{n}]$ s.t. $\forall h \in]0, \frac{1}{n}]$

$$\left| \frac{f(x+h) - f(x)}{h} \right| < n \}$$

Exercise $\quad D \subseteq \bigcup_n C_n$

and $\forall n,\ C_n \subseteq D$

I prove that C_n is closed

I take $f \in \overline{C_n} \leftarrow$ closure of C_n in $\mathcal{C}([0,1])$

$\exists (f_k)_k$ in C_n such that $f_k \to f$ uniformly

for each f_k $\exists x_k$ s.t. $x_k \in [0, 1-\frac{1}{n}]$

and $\forall h \in]0, \frac{1}{n}]$ $\left| \frac{f_k(x_k+h) - f_k(x_k)}{h} \right| < n$

in particular passing to subsequence $x_k \to \bar{x} \in [0, 1-\frac{1}{n}]$.

Take $\varepsilon > 0$, take $h \in]0, \frac{1}{n}]$

it is possible to choose $\bar{k}$ such that, for k m,

$\| f_k - f \|_\infty \leq \frac{\varepsilon h}{4}$, $\quad |f(x_k) - f(\bar{x})| \leq \frac{\varepsilon h}{4}$,

$|f(x_k+h) - f(\bar{x}+h)| \leq \frac{\varepsilon h}{4}$

$$|f(\bar{x}+h) - f(\bar{x})| \leq \overbrace{|f(\bar{x}+h) - f(x_k+h)|}^{\leq \frac{\varepsilon h}{4}} + \overbrace{|f(x_k+h) - f_k(x_k+h)|}^{\leq \|f - f_k\|_\infty \leq \frac{\varepsilon h}{4}}$$
$$+ |f_k(x_k+h) - f_k(x_k)| +$$
$$\underbrace{|f_k(x_k) - f(x_k)|}_{\leq \|f - f_k\|_\infty \leq \frac{\varepsilon h}{4}} + \underbrace{|f(x_k) - f(\bar{x})|}_{\frac{\varepsilon h}{4}}$$

$f_k \in C_n$

$\Downarrow$

$\leq hn$

$|f(\bar{x}+h) - f(\bar{x})| \leq \varepsilon h + hn = (n+\varepsilon) h$

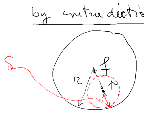
$$\boxed{\left| \frac{f(\bar{x}+h) - f(\bar{x})}{h} \right| \leq n + \varepsilon} \Rightarrow \left| \frac{f(\bar{x}+h) - f(\bar{x})}{h} \right| \leq n$$

$\Downarrow$

$f \in C_n$

to end I have to prove that C_n has empty interior,

by contradiction.



$B(f, r) \subseteq C_n$

$\{ g \in \mathcal{C}([0,1]) : \|f - g\|_\infty < r \}$

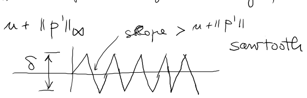
Weierstrass th. $\quad \exists p$ polynomial function s.t.

$\|p - f\|_\infty < r \qquad \delta = r - \|f - p\|_\infty \qquad B(p, \delta) \subseteq C_n$

I consider $g \in C([0,1])$ such that $\|g\|_\infty < \delta$

and g is differentiable from the right at every point

and $g'_+(x) > n + \|p'\|_\infty$

$g =$ 

consider $p + g \in B(p, \delta) \subseteq C_n$

but $(p+g)'_+(x) > n \quad \forall x \Rightarrow p + g \notin C_n$