Lesson 2   march 4th

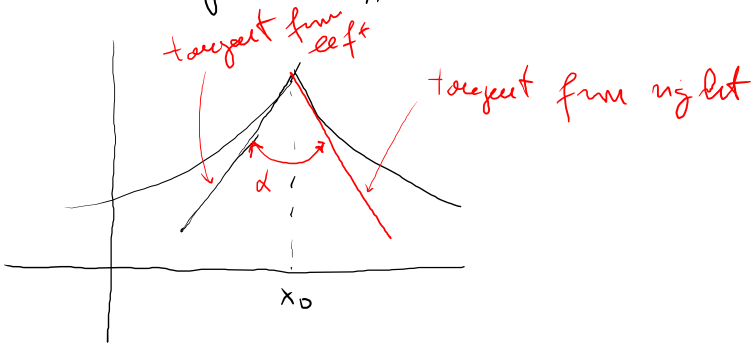
- $\exists f \in \mathcal{C}(\mathbb{R})$ and f nowhere diff.
- The set of such functions is "big"

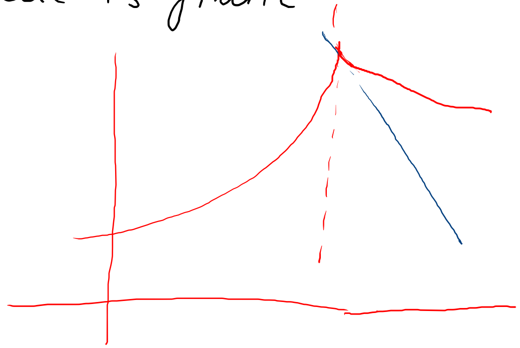
¿ How many angles or cusps are there ?

def. let $f : I \to \mathbb{R}$ I int, $x_0 \in I$, f continuous at x_0

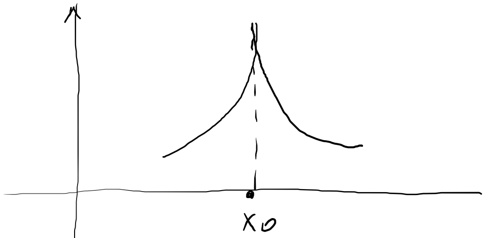
x_0 is "angle" if $\exists \overset{f'_-(x_0)}{\lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}}$, $\exists \overset{f'_+(x_0)}{\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}}$

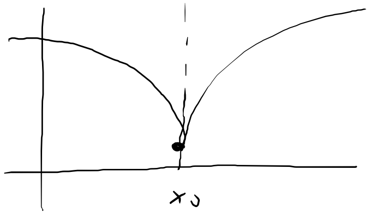
they are different and at least one is finite





x_0 is "cusp" if the two above limits are infinite and different.





Theorem let $f : I \to \mathbb{R}$ f continuous
open interval

$C = \{x \in I : x \text{ is angle or cusp}\}$

then C is finite or countable.

proof.

$A = \{x \in I : x \text{ is angle or cusp and } f'_+(x) > f'_-(x)\}$

$B = \{x \in I : \text{" " " " } f'_+(x) < f'_-(x)\}$

I show that A is finite or countable

Suppose $x_0 \in A$ so that $f'_+(x_0) > f'_-(x_0)$

then $\exists\ r \in \mathbb{Q}$ s.t. $\underbrace{f'_-(x_0) < r}_{} < f'_+(x_0)$

moreover

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} = f'_-(x_0) < r$$

$\Rightarrow \exists\ s \in \mathbb{Q}$ $s < x_0$ s.t. $\forall\ y \in]s, x_0[$

$$\frac{f(y) - f(x_0)}{y - x_0} < r \qquad (*)$$

similarly $\exists\ t \in \mathbb{Q}$ s.t. $t > x_0$ and $\forall z \in]x_0, t[$

$$\frac{f(z) - f(x_0)}{z - x_0} > r \qquad (**)$$

I have a $\Phi : A \to \mathbb{Q}^3$

$x \longmapsto (r, s, t)$ $\quad s < x < t$

and $(*)$ and $(**)$ hold.

I claim that Φ is injective

by contradiction $\exists\ x_1, x_2 \in A$ s.t. $x_1 \longrightarrow (r, s, t)$
$\qquad \qquad \qquad \qquad \qquad \qquad ||$
$\qquad \qquad \qquad \qquad \qquad x_2 \longrightarrow (r, s, t)$

then $s < x_1 < x_2 < t$

and $(*)$ is valid with $x_2 = x_0$, $x_1 = y$

so $$\frac{f(x_1) - f(x_2)}{x_1 - x_2} < r$$

and $(**)$ is valid with $x_1 = x_0$, $x_2 = z$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > r$$

impossible

Lebesgue's differentiation theorem

Th let $f: I \to \mathbb{R}$ I interval.

If f is monotone then f is almost everywhere differentiable

proof (in the case of continuous functions) (Kolmogoroff-Fomin)

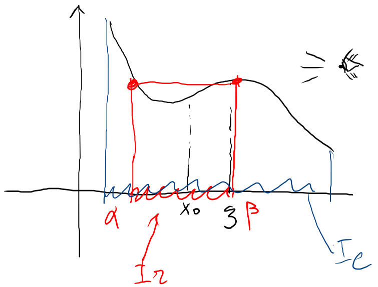
def. let $f: [a,b] \to \mathbb{R}$ continuous function

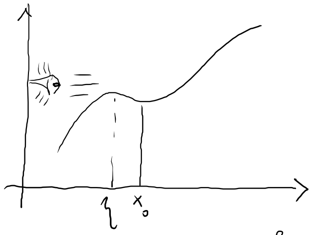
let $x_0 \in [a,b[$ x_0 is called invisible from the right

if $\exists\, \xi \in]x_0, b[$ s.t. $f(x_0) < f(\xi)$

similarly, if $x_0 \in]a,b]$, x_0 is invisible from the left

if $\exists\, \eta \in]a, x_0[$ s.t. $f(\eta) > f(x_0)$





Lemma let f as before, let $I_r = \{$ invisible from the right point $\}$

then I_r is open, $I_r = \bigcup_k]\alpha_k, \beta_k[$ pairwise disjoint

and $f(\alpha_k) \le f(\beta_k)$.

similarly $I_\ell = \{$ invisible points from left $\}$

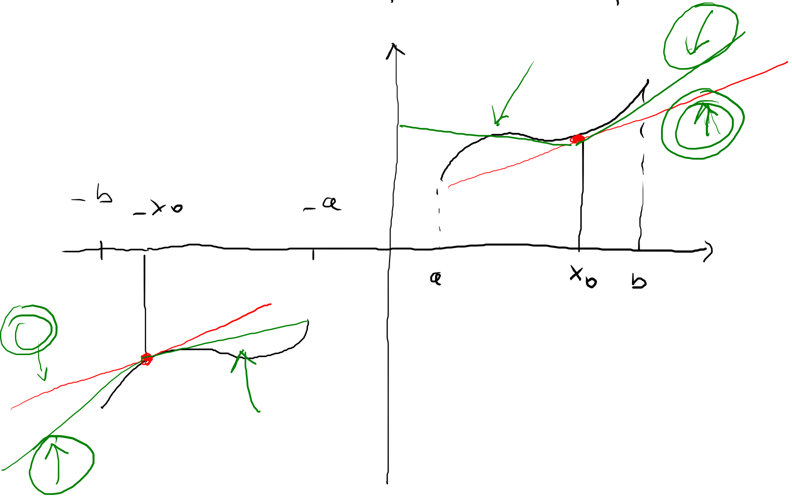
I_ℓ open, $I_\ell = \bigcup_k]\alpha'_k, \beta'_k[$ pairwise disjoint

$f(\alpha'_k) \geq f(\beta'_k)$

proof (exercise)

Rem. consider $f : [a,b] \to \mathbb{R}$ and define $\tilde{f} : [-b,-a] \to \mathbb{R}$

$$\tilde{f}(x) = -f(-x)$$



$$(*) \quad f'(x_0) = \tilde{f}'(-x_0)$$

$$f'_+(x_0) = \tilde{f}'_-(-x_0)$$

$$(**)$$

$$f'_-(x_0) = \tilde{f}'_+(-x_0)$$

$$D^+ f(x_0) = D^- \tilde{f}(-x_0)$$

$$D_+ f(x_0) = D_- \tilde{f}(-x_0)$$

$$(***) \quad D^- f(x_0) = D^+ \tilde{f}(-x_0)$$

$$D_- f(x_0) = D_+ \tilde{f}(-x_0)$$

suppose f continuous, $f : [a,b] \to \mathbb{R}$ increasing.

Claim if I'm able to prove

1) for almost all x_0 in $[a,b]$, $D^+ f(x_0) < +\infty$

2) for almost all $x_0 \in [a,b]$, $D^+ f(x_0) \leq D_- f(x_0)$

then the conclusion of the theorem follows

in fact we know that $D_- f(x_0) \leq D^- f(x_0)$ $\forall x_0$

$D_+ f(x_0) \leq D^+ f(x_0)$ $\forall x_0$

2) is valid for f monotone functions

in particular is valid also for $\tilde{f}$

$$\begin{array}{ccc} D^+ \tilde{f}(-x_0) & \leq & D_- \tilde{f}(-x_0) \\ \parallel & & \parallel \\ D^- f(x_0) & & D_+ f(x_0) \end{array}$$

2) $\Rightarrow$ $D^- f(x_0) \leq D_+ f(x_0)$

I can conclude

$$D^+ f(x_0) \leq D_- f(x_0) \leq D^- f(x_0) \leq D_+ f(x_0) \leq D^+ f(x_0)$$

they are all equal!

and from 1) they are $< +\infty$ (and since increasing all are ≥ 0)

I denote by $K = \{ x \in]a,b[: D^+ f(x) = +\infty \}$

I want to prove that $\lambda(K) = 0$ (λ is the Lebesgue measure)

let $x_0 \in K$ $\quad D^+ f(x_0) = +\infty \Rightarrow \underbrace{\operatorname{suplim}_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}}_{\boxed{\inf_{t>0}} \left(\sup_{x_0<x<x_0+t} \frac{f(x)-f(x_0)}{x-x_0} \right) \boxed{= +\infty}} = +\infty$

$\forall t>0 \quad \sup_{x_0<x<x_0+t} \frac{f(x)-f(x_0)}{x-x_0} = +\infty$

For all $\underline{\text{fixed } C>0}$ $\quad \exists\, \xi \in]x_0, b[\; : \; \frac{f(\xi)-f(x_0)}{\xi - x_0} > C$

$\Rightarrow \quad f(\xi) - f(x_0) > C(\xi - x_0)$

$\Rightarrow \quad f(\xi) - C\xi > f(x_0) - Cx_0 \quad$ for $\quad \xi \in]x_0, b[$

$\Rightarrow$ x_0 is invisible from the right for the function

$\phi :]a,b[\to \mathbb{R} \qquad \phi(y) = f(y) - Cy$

$K \subseteq I_c = \{$invisible from the right for $\phi\} = \bigcup_k]\alpha_k, \beta_k[$

with $\underbrace{\phi(\alpha_k) \le \phi(\beta_k)}$

$\Updownarrow$

$f(\alpha_k) - C\alpha_k \le f(\beta_k) - C\beta_k$

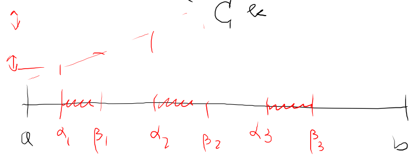
$\Downarrow$

$\beta_k - \alpha_k \le \frac{f(\beta_k) - f(\alpha_k)}{C}$

$\lambda(I_c)$

so $\sum_k (\beta_k - \alpha_k)$

$\le \frac{1}{C} \sum_k f(\beta_k) - f(\alpha_k)$



f increasing

$\le \frac{1}{C} f(b) - f(a)$

$K \subseteq \bigcap_{C>0} I_c \qquad \lambda(I_c) \le \frac{f(b)-f(a)}{C} \left(\xrightarrow{C \to +\infty} 0 \right)$

$\Rightarrow \lambda(K) = 0$

Let's prove 2) (We have to prove that $D^+f(x) \le D_-f(x)$ almost everywhere)

equivalent $\boxed{H = \{x \in]a,b[\; : \; D^+f(x) > D_-f(x)\}}$ has measure 0

fix $r, q > 0$, $r, q \in \mathbb{Q}$ with $0 < r < q$

denote by $K_{r,q} = \{x \in]a,b[\; : \; D_-f(x) < r < q < D^+f(x)\}$

$$H = \bigcup_{\substack{r,q \in \mathbb{Q} \\ 0<r<q}} K_{r,q}$$

↑ countable union

It is sufficient to prove $\lambda(K_{r,q}) = 0$

<u>Lemma</u> Let $A \subseteq]a,b[$ and let $\rho \in]0,1[$

Suppose that for all $]\alpha,\beta[\subseteq]a,b[$,

$$\lambda^*(A \cap]\alpha,\beta[) \le \rho(\beta-\alpha)$$

Then $\lambda^*(A) = 0$ (and then A is measurable and $\lambda(A) = 0$)

(Exercise)

Take $x_0 \in K_{c,C} \cap]\alpha,\beta[$ remember $D_- f(x_0) < c$

and $D_+ f(x_0) > C$

$D_- f(x_0) < c$ so that

$$\sup_{t>0}\left(\inf_{x_0-t<x<x_0} \frac{f(x)-f(x_0)}{x-x_0}\right) < c$$

$\forall t$ $\inf_{x_0-t<x<x_0} \frac{f(x)-f(x_0)}{x-x_0} < c$ $\exists\, \eta \in]x_0-t, x_0[$ s.t

$$\frac{f(\eta)-f(x_0)}{\eta-x_0} < c$$

$$f(\eta)-f(x_0) > c(\eta-x_0)$$

$$\boxed{f(\eta) - c\eta > f(x_0) - c x_0 \quad \text{with} \quad \eta < x_0}$$

x_0 is $\underbrace{\text{invisible from the left}}$ for $\Phi:]\alpha,\beta[\to \mathbb{R}$

$y \mapsto f(y) - c\eta$

$\|$

previous deput $\longrightarrow$ $\bigcup_k]\alpha_k,\beta_k[$ with $f(\alpha_k) - c\alpha_k \geq f(\beta_k) - c\beta_k$

$\Updownarrow$

$$\boxed{c(\beta_k - \alpha_k) \geq f(\beta_k) - f(\alpha_k)}$$

$\exists\, \bar{k} :$ $x_0 \in]\alpha_{\bar{k}}, \beta_{\bar{k}}[$, $D^+ f(x_0) > C$

$\exists\, \xi > x_0$ s.t $\frac{f(\xi)-f(x_0)}{\xi - x_0} > C$

$\Rightarrow$ $f(\xi) - f(x_0) > C(\xi - x_0)$

$\Rightarrow$ $f(\xi) - C\xi > f(x_0) - C x_0$

$\Rightarrow$ x_0 $\underbrace{\text{invisible from right}}$ for $\Psi:]\alpha_{\bar{k}}, \beta_{\bar{k}}[\to \mathbb{R}$

$\Psi(y) = f(y) - Cy$

previous dey $\longrightarrow$ $\bigcup_\gamma]\alpha_{\bar{k},\gamma}, \beta_{\bar{k},\gamma}[$ with $f(\beta_{\bar{k},\gamma}) - C\beta_{\bar{k},\gamma} \geq f(\alpha_{\bar{k},\gamma}) - C\alpha_{\bar{k},\gamma}$

$$f(\beta_{\bar{k},\gamma}) - f(\alpha_{\bar{k},\gamma}) \geq C(\beta_{\bar{k},\gamma} - \alpha_{\bar{k},\gamma})$$

$$(\beta_{\bar{k},\gamma} - \alpha_{\bar{k},\gamma}) \leq \frac{1}{C}\left(f(\beta_{\bar{k},\gamma}) - f(\alpha_{\bar{k},\gamma})\right)$$

$$(K_{c,C} \cap]\alpha,\beta[) \subseteq \bigcup_k \left(\bigcup_\gamma]\alpha_{k,\gamma}, \beta_{k,\gamma}[\right)$$

inv from left (for k)

inv. frm right (for γ)

$$\lambda\left(\bigcup_k\left(\bigcup_\gamma (]\quad[)\right)\right) = \sum_k \left(\sum_\gamma (\beta_{k,\gamma} - \alpha_{k,\gamma})\right)$$

$\beta_{k,\gamma} - \alpha_{k,\gamma} \leq \frac{1}{C} f(\beta_{k,\gamma}) - f(\alpha_{k,\gamma})$

$$\leq \sum_k \left(\underbrace{\sum_\gamma \frac{1}{C}\left(f(\beta_{k,\gamma}) - f(\alpha_{k,\gamma})\right)}_{\leq \frac{1}{C}(f(\beta_k) - f(\alpha_k))}\right)$$

$$\leq \sum_k \frac{1}{C}\left(f(\beta_k) - f(\alpha_k)\right),$$

$$\leq \sum_k \frac{c}{C}(\beta_k - \alpha_k) \leq \frac{c}{C}(\beta - \alpha)$$

Conclusion

$\lambda^*(K_{c,C} \cap]\alpha\beta[) \leq \lambda(\bigcup(\bigcup)) \leq \frac{c}{C}(\beta - \alpha)$ by the lemm $\Rightarrow \lambda(K_{c,C}) = 0$