Lesson 2 march 8^{th}

Lebesgue's diff theorem: if $f : [a,b] \to \mathbb{R}$ monotone, then f is a.e. differentiable

Exercise (Fubini's th. on differentiation of series)

Th. Let $(f_n)_n$ sequence of increasing functions $f_n : [a,b] \to \mathbb{R}$. Suppose that $\forall x \in [a,b]$, the series $\sum_n f_n(x)$ is convergent (with sum $s(x)$) pointwise.

then
1) s is a.e. differentiable
2) the series $\sum_n f_n'(x)$ is a.e. convergent
3) the sum of $\sum_n f_n'(x)$ is a.e. $s'(x)$

proof. It is not restrictive to consider $f_n(x) \geq 0$ $\forall n, \forall x$
(otherwise set $g_n(x) := f_n(x) - f_n(a)$ and work with g_n)

We set $s_n(x) = \sum_{j=1}^{n} f_j(x)$ so $s_n : [a,b] \to \mathbb{R}$
s_n is increasing and positive
and $s_n(x) \to s(x)$

so also s is positive and increasing $\Rightarrow$ Lebesgue's $\Rightarrow$ s is diff. a.e.

Remark that

$$s(x) = \underbrace{s_n(x)}_{= \sum_{j=1}^{n} f_j} + \underbrace{r_n(x)}_{= \sum_{j=n+1}^{+\infty} f_j}$$

so also r_n is increasing and diff. a.e.

obviously $s'(x) = s_n'(x) + r_n'(x)$ a.e.

moreover $s_n'(x) = \sum_{j=1}^{n} f_j'(x) \leq \sum_{j=1}^{n} f_j'(x) + \underbrace{f_{n+1}'(x)}$

remember that f_{n+1} is increasing and a derivative of an increasing funct is positive ≥ 0

$$\Rightarrow \quad s_n'(x) \leq \underbrace{s_{n+1}'(x) \leq s'(x)}$$

because $s'(x) = s_{n+1}'(x) + r_{n+1}'(x)$

notice r_{n+1} increasing and then $r_{n+1}' \geq 0$

conclusion $s_n'(x) \leq s'(x)$

this increasing (in n) sequence, so it is a.e. convergent.

(this is the point 2))

It remains to prove that $\sum_n f_n'(x)$ is equal to $s'(x)$ a.e.

(for the moment $\sum_n f_n'(x) \leq s'(x)$)

$$\sum_n f_n'(x) = \lim_n s_n'(x) = \lim_n \sum_{j=1}^{n} f_j'(x)$$

for a.all $x \in [a,b]$, this is an increasing (in n) sequence

it is sufficient to prove that $\exists (n_k)_k$ such that $\boxed{\lim_k s_{n_k}'(x) = s'(x)}$

Consider $r_n(x) = s(x) - s_n(x)$ we have $r_n(b) = s(b) - s_n(b)$

and $0 \leq r_n(a) \leq r_n(x) \leq r_n(b)$

moreover $\lim_n r_n(b) = 0$ $\quad (s_n(b) \to s(b))$

Consequently $\exists n_k$ s.t. $\sum_k r_{n_k}(b) < +\infty$ $\left(\text{e.g. } r_{n_k}(b) \leq \frac{1}{2^k}\right)$

so $\sum_k \underbrace{r_{n_k}(x)}_{g_k(x)} < +\infty$ $\forall x \in [a,b]$

g_k are increasing, positive and $\sum_k g_k(x) < +\infty$ $\forall x \in [a,b]$

Apply the first part of the theorem, we have that $\sum_k g_k'(x) < +\infty$ a.e.

this means that $\lim_k g_{n_k}'(x) = 0$ a.e.

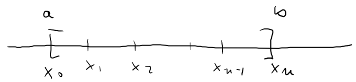
but $g_{n_k}'(x) = s'(x) - s_{n_k}'(x) \xrightarrow{k} 0$ for a.e. x

CVD.

Functions with bounded variation

def. Let $[a,b]$ bounded closed interval in $\mathbb{R}$

let $\Delta = \{ a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b \}$ finite number of points in $[a,b]$ containing a and b



we call Δ "subdivision" of $[a,b]$.

def let $f : [a,b] \to \mathbb{R}$

we define $V(f,\Delta) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|$ Variation of f w.r.t. the subdivision Δ

$BV([a,b])$

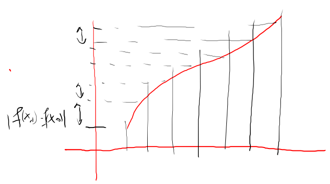
def. f is of bounded variation on $[a,b]$ if

$$\sup_{\Delta \in \mathcal{D}} V(f,\Delta) = V_a^b(f) < +\infty$$

where $\mathcal{D} = \{$ subdivisions of $[a,b]\}$

this quantity is called total variation of f

Ex.



f increasing

$V_a^b(f) = f(b) - f(a)$

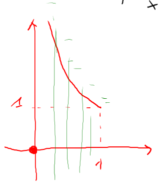
since $\forall \Delta = \{x_0 < \dots < x_n\}$

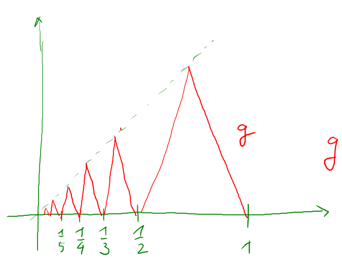
we have $\sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| =$

$= \sum_{j=1}^{n} f(x_j) - f(x_{j-1}) = f(b) - f(a)$

Ex. $f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \frac{1}{x} & \text{for } x \in]0,1] \end{cases}$ $f : [0,1] \to \mathbb{R}$

$f \notin BV([0,1])$





$g \in \mathcal{C}([0,1])$

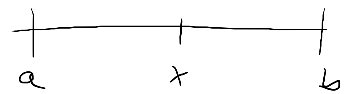
$g \notin BV([0,1])$

try as exercise

First properties of BV functions

i) if $f \in BV([a,b])$ then f is bounded

$\forall x \in [a,b]$ $\quad |f(x)| \le |f(a)| + \underbrace{|f(x) - f(a)| + |f(b) - f(x)|}_{V(f,\tilde{\Delta})} \qquad \tilde{\Delta} = \{a, x, b\}$



$|f(x)| \le |f(x) - f(a)| + |f(a)| -$

$\le |f(a)| + |f(x) - f(a)| + |f(x) - f(b)| \le |f(a)| + V_a^b(f)$

ii) $f \in BV([a,b])$ $\quad \alpha \in \mathbb{R} \Rightarrow \alpha f \in BV([a,b])$

and $\quad V_a^b(\alpha f) = |\alpha| V_a^b(f)$

iii) $f, g \in BV([a,b])$, then $f + g \in BV[a,b]$ and

$V_a^b(f+g) \le V_a^b(f) + V_a^b(g)$

$BV([a,b])$ is a Vector space.

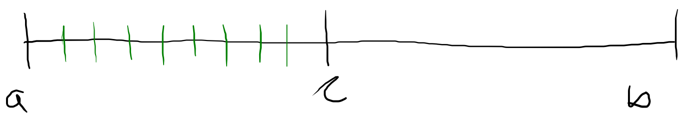
Ex. setting $\|f\|_{BV} = |f(a)| + V_a^b(f)$ is a norm on BV

BV is complete w.r.t. $\|f\|_{BV}$ $\rightarrow$ $\boxed{\text{and BV is a Banach space}}$

iv) let $c \in [a,b]$ let $f \in BV([a,b])$

then $f_{|[a,c]} \in BV([a,c])$ and $f_{|[c,b]} \in BV([c,b])$

and $V_a^b(f) = V_a^c(f) + V_c^b(f)$



v) let $f \in BV([a,b])$ consider $V_a^x(f)$ as a function on $[a,b]$, the function $x \longrightarrow V_a^x(f)$ is increasing

$$V_a^b(f) = V_a^x(f) + \underbrace{V_x^b(f)}_{\geq 0}$$

$$V_a^{x_2}(f) - V_a^{x_1}(f) = V_{x_1}^{x_2}(f)$$

vi) let $f \in BV([a,b])$ and suppose that f is continuous at the point $\bar{x} \in [a,b]$

then the function $x \longmapsto V_a^x(f)$ is continuous at the point $\bar{x}$

<u>exercise</u>

Theorem $f : [a,b] \to \mathbb{R}$

f is a BV-function iff f is the difference of two increasing functions

proof. if $f = g_1 - g_2$ with g_1, g_2 increasing

then $g_1, g_2 \in BV$ then $f \in BV$

conversely let $f \in BV$ consider $f(x) = \underbrace{V_a^x(f)}_{\text{increasing}} - \left(V_a^x(f) - f(x)\right)$

but also $x \longmapsto V_a^x(f) - f(x)$ is increasing

in fact $\forall\, x_2, x_1 \in [a,b]$, $x_1 < x_2$ $\quad f(x_2) - f(x_1) \le |f(x_2) - f(x_1)| \le V_{x_1}^{x_2}(f) = V_a^{x_2}(f) - V_a^{x_1}(f)$

so $\quad f(x_2) - f(x_1) \le V_a^{x_2}(f) - V_a^{x_1}(f)$

consequently $\quad V_a^{x_1}(f) - f(x_1) \le V_a^{x_2}(f) - f(x_2)$

C.V.D.

Corollary if $f \in BV([a,b])$ then f is a.e. differentiable

proof f = difference of $\underbrace{\text{increasing functs}}_{\text{differentiable by Lebesgue's th.}}$

Corollary if $f \in BV([a,b])$ then

$\forall\, \bar{x} \in]a,b[$, $\exists \lim_{x \to \bar{x}^-} f(x)$, $\exists \lim_{x \to \bar{x}^+} f(x)$

The integral function of a L^1 function

Th. let $f \in L^1(a,b)$ (= integrable Lebesgue functions in $[a,b]$)

We define $F(x) = \int_{[a,x]} f = \int_a^x f(t)\,dt$

F is the integral-function of f

then
1) F is uniformly continuous on $[a,b]$
2) F is $BV([a,b])$ and $V_a^b(F) = \|f\|_{L^1}$

proof. $\forall \bar{x} \in [a,b]$, F is continuous in $\bar{x}$

suppose $(x_n)_n$ is a sequence in $[a,b]$ s.t. $x_n \to \bar{x}$

then consider $f(x)\chi_{[a,x_n]}(x) = f_n(x)$

I have that $f_n \in L^1(a,b)$

$|f_n| \le |f|$ in $[a,b]$

so that $\int_{[a,b]} f_n \longrightarrow \int_{[a,b]} f\chi_{[a,\bar{x}]}(x)$ (dominated convergence th.)

since $f_n \to f \cdot \chi_{[a,\bar{x}]}$ pointwise

this is $\lim_n F(x_n) = F(\bar{x})$ for all $x_n \to \bar{x}$

$\Rightarrow$ F is continuous at $\bar{x}$. Since $[a,b]$ closed and bounded F is uniformly cont.

F is BV infact let $\Delta = \{a = x_0 < x_1 < \dots < x_{n-1} < x_n = b\}$

for all Δ

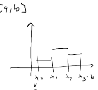
$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{n} \left| \int_a^{x_j} f(t)\,dt - \int_a^{x_{j-1}} f(t)\,dt \right| = \sum_{j=1}^{n} \left| \int_{x_{j-1}}^{x_j} f(t)\,dt \right|$$

$$\le \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |f(t)|\,dt = \int_a^b |f(t)|\,dt = \|f\|_{L^1}$$

$\Rightarrow$ $F \in BV$ and $V_a^b(F) \le \|f\|_{L^1}$

To show the equality: step functions in $[a,b]$

$$\sum_{j=1}^{n} \alpha_j \chi_{[x_{j-1},x_j[}(x)$$



step functions are dense in L^1

I take $\operatorname{sgn}(f) = \begin{cases} 1 & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) = 0 \\ -1 & \text{if } f(x) < 0 \end{cases}$

$\operatorname{sgn}(f) \in L^1(a,b)$ and $\exists (\sigma_n)_n$ σ_n step functions

s.t. $\sigma_n \to \operatorname{sgn} f$ in L^1

it is not restrictive to consider $|\sigma_n| \le 1$ and

$\sigma_n(x) \to \operatorname{sgn} f(x)$ for almost all $x \in [a,b]$.

I know that

$|f(x)| = (\operatorname{sgn} f(x)) \cdot f(x) = \lim_n \sigma_n(x) \cdot f(x)$ for a.a. points in (a,b)

and $|\sigma_n(x)| \le 1$

so $|\sigma_n(x) \cdot f(x)| \le |f(x)|$

I have a sequence $(\sigma_n(x) f(x))$ which goes to $|f(x)|$ and it is dominated by $|f(x)| \in L^1$

$\Rightarrow$ (dominated convergence theorem)

$$\lim_n \int_a^b \sigma_n(x) f(x)\,dx = \int_a^b |f(x)|\,dx = \|f\|_{L^1}$$

but $\left| \int_a^b \sigma_n(x) f(x)\,dx \right|$ $\qquad \sigma_n(x) = \sum_{j=1}^{k} \alpha_{n,j} \chi_{[x_{n,j}, x_{n,j+1}[}(x)$

$\qquad\qquad |\alpha_{n,j}| \le 1$

$$\wedge\!\!\!\wedge$$

$$\left| \int_a^b \sum_{j=1}^{k} \alpha_{n,j} \chi_{[x_{n,j-1}, x_{n,j}[}(x) f(x)\,dx \right|$$

$$\wedge\!\!\!\wedge$$

$$\sum_{j=1}^{k} \underbrace{|\alpha_{n,j}|}_{\le 1} \left| \int_{x_{n,j-1}}^{x_{n,j}} f(t)\,dt \right| \le \sum_{j=1}^{k} 1 \int_{x_{n,j-1}}^{x_{n,j}} f(t)\,dt \le \sum V_{x_{n,j-1}}^{x_{n,j}}(F)$$

$$\le V_a^b(F)$$

conclusion $V_a^b(F) \ge \|f\|_{L^1}$

$\Rightarrow$ $V_a^b(F) = \|f\|_{L^1}$