Lesson 5 march 15^{th}

def $AC([a,b]) \ni f$ $\quad f : [a,b] \to \mathbb{R}$ (or $\mathbb{C}$)

$\forall \varepsilon > 0,\ \exists \delta > 0 : \ \forall (]\alpha_k, \beta_k[)_k$ pairwise disjoint in $]a,b[$

if $\sum_k (\beta_k - \alpha_k) < \delta$ then $\sum_k |f(\beta_k) - f(\alpha_k)| < \varepsilon$

Ex. let $f \in L^1(a,b)$ then $F(x) := \int_a^x f(t)\,dt$

$F \in AC([a,b])$

- $AC([a,b])$ vector space
- $AC([a,b]) \subseteq \mathcal{C}([a,b])$ $\longleftarrow$ continuous functions
- $AC([a,b]) \subseteq BV([a,b])$
- if $f \in AC([a,b])$ then also $x \longmapsto V_a^x(f)$ is in $AC([a,b])$

consequently $f \in AC([a,b]) \Rightarrow f$ difference of increasing AC-functions

Problem: we know that the integral function of an integrable function is $AC([a,b])$
¿ is the converse true ? YES (a part a constant)

Th Let $f \in AC([a,b])$.

[We know that f is $BV([a,b])$, so that f is a.e. differentiable and $f' \in L^1(a,b)$.]

Then, $\forall x \in [a,b]$, $\quad f(x) = f(a) + \int_a^x f'(t)\,dt$

Lemma 1. Let $F \in AC([a,b])$ and F is increasing

If $F'(x) = 0$ a.e.

Then F is constant.

proof. Let $A = \{ x \in \,]a,b[\; : \; F'(x) \neq 0 \}$

we have $\lambda(A) = 0$

Lebesgue's measure

f is $AC([a,b])$ so that take $\varepsilon > 0$ and we have $\delta \ldots$

from the definition of AC

A is a set of measure 0 then $\exists \, (]\alpha_k, \beta_k[)_k$ - p. disjoint

s.t. $A \subseteq \bigcup_k \,]\alpha_k, \beta_k[$

from the regularity of Lebesgue's meas.

and $\sum_k (\beta_k - \alpha_k) < \delta$

So

$$F(A) \subseteq \bigcup_k F(]\alpha_k, \beta_k[)$$

$$\lambda(F(A)) \leq \lambda\Big(\bigcup_k \underbrace{F(]\alpha_k, \beta_k[)}_{\subseteq \,]F(\alpha_k), F(\beta_k)[}\Big)$$

F is increasing

$$\leq \sum_k |F(\beta_k) - F(\alpha)| < \varepsilon$$

conclusion $\lambda(F(A)) < \varepsilon \quad \forall \varepsilon > 0 \implies \lambda(F(A)) = 0$

Let now $B = \{ x \in \,]a,b[\; : \; F'(x) = 0 \}$

$$x_0 \in B \implies F'(x_0) = 0 = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0}$$

$\forall \varepsilon > 0, \ \exists r > 0 : \quad \forall x \in \,]x_0 - r, x_0 + r[\, \setminus \{x_0\},$

$$\left| \frac{F(x) - F(x_0)}{x - x_0} \right| < \varepsilon$$

In particular $\forall x \in \,]x_0, x_0 + r[\qquad F(x) - F(x_0) < \varepsilon(\underbrace{x - x_0}_{>0})$

In particular $\varepsilon x_0 - F(x_0) < \varepsilon x - F(x)$ if $x \in \,]x_0, x_0 + r[$

this implies that x_0 is invisible from the right for the function $x \longmapsto \varepsilon x - F(x)$

so let $I_{r,\varepsilon}$ the set of inv. points from right for $x \longmapsto \varepsilon x - F(x)$

$= \bigcup_k \,]\alpha_k, \beta_k[$ with $\varepsilon \alpha_k - F(\alpha_k) \leq \varepsilon \beta_k - F(\beta_k)$

and $B \subseteq I_{r,\varepsilon} \implies F(B) \subseteq F(I_{r,\varepsilon})$

$$\lambda(F(I_{r,\varepsilon})) = \lambda\Big(F\big(\bigcup_k \,]\alpha_k, \beta_k[\big)\Big)$$

$$\leq \sum_k F(\beta_k) - F(\alpha_k)$$

F is increasing

$$F(\beta_k) - F(\alpha_k) < \varepsilon(\beta_k - \alpha_k)$$

p. disjoint

$$\leq \sum_k \varepsilon(\beta_k - \alpha_k) = \varepsilon \sum_k (\beta_k - \alpha_k)$$

$$\leq \varepsilon(b - a)$$

F' does not exist or it exists and it is $\neq 0$

conclusion $\lambda(F(B)) \leq \lambda(F(I_{r,\varepsilon})) \leq \varepsilon(b-a) \quad \forall \varepsilon > 0$

$\implies \boxed{\lambda(F(B)) = 0}$

$A = \{ x : F'(x) \neq 0 \} \qquad \lambda(F(A)) = 0$

$B = \{ x : F'(x) = 0 \} \qquad \lambda(F(B)) = 0$

but $A \cup B = \,]a,b[$ so $\lambda(F(]a,b[)) = 0$

by calculus $\longrightarrow$ (we know that F is continuous

$\implies$ F is constant

QED

proof of the Theorem

It is not restrictive to suppose that f is increasing

consider $a \leq x_1 < x_2 \leq b$

we have $\displaystyle\int_{x_1}^{x_2} \underbrace{f'(t)}\,dt \leq f(x_2) - f(x_1)$ ("lemma using Fatou")

we know that $f' \geq 0$, and $f' \in L^1$ (lemma of last week)

$$\int_a^{x_2} f' - \int_a^{x_1} f'$$

so that $$\int_a^{x_2} f'(t)\,dt - f(x_2) \leq \int_a^{x_1} f'(t)\,dt - f(x_1)$$

i.e. the function $\boxed{x \longmapsto \int_a^x f'(t)\,dt - f(x)}$ is increasing

moreover it is also an AC function

(yes, since f is AC by hypothesis and $x \mapsto \int_a^x f'(t)\,dt$ is an integral function)

I compute the derivative

$$\left(\int_a^x f'(t)\,dt - f(x) \right)' = f'(x) - f'(x) = 0$$

We apply the lemma : the function $x \longmapsto \int_a^x f'(t)\,dt - f(x)$ is a constant

so that $\displaystyle\int_a^x f'(t)\,dt - f(x) = C$ but immediately $C = -f(a)$

finally $\displaystyle f(x) = f(a) + \int_a^x f'(t)\,dt$

QED

Integration by parts for AC functions.

Th Let $f, g \in AC([a,b])$.

then $f \cdot g$ is $AC([a,b])$, for almost all $x \in [a,b]$

$$(f \cdot g)'(x) = f(x) g'(x) + f'(x) g(x)$$

and

$$\int_a^b f'(t) g(t) dt = f(b)g(b) - f(a)g(a) - \int_a^b f(t) g'(t) dt$$

why this function is in L^1? $f' \in L^1$ and g is in L^∞ (g it is continuous)

proof. ⊛ is not difficult,

because f is a.e. diff. g is a.e. diff.

so f and g are diff in a set of measure $b-a$

in a point in which both are diff, we use the elementary result of calculus

it remains to prove that $f \cdot g$ is AC.

take $(]\alpha_k, \beta_k[)_k$ we have

$$\sum_k |f(\beta_k) g(\beta_k) - f(\alpha_k) g(\alpha_k)| \leq$$

$$\leq \sum_k |f(\beta_k) g(\beta_k) - f(\beta_k) g(\alpha_k)| + \sum_k |f(\beta_k) \cdot g(\alpha_k) - f(\alpha_k) g(\alpha_k)|$$

$$\leq \max_{[a,b]} |f| \cdot \sum_k |g(\beta_k) - g(\alpha_k)| + \max_{[a,b]} |g| \sum_k |f(\beta_k) - f(\alpha_k)|$$

δ_1 $< \frac{\varepsilon}{(\max |f|)+1}$ $< \frac{\varepsilon}{(\max |g|)+1}$ δ_2

Remark consider $W^{1,1} = \{ f \in L^1(a,b) : \exists g \in L^1(a,b) :$

$\forall \varphi \in \mathcal{C}^1_0(]a,b[),$

$\int_a^b f \cdot \varphi' = - \int_a^b g \cdot \varphi \}$

Sobolev space

$\mathcal{C}^1$ functions with compact support in $]a,b[$

is the closure of $\{\varphi(x) \neq 0\}$

Verify that $AC([a,b]) \subseteq W^{1,1}$

$F \in AC \Rightarrow F \in L^1(a,b)$ and $\int_a^b F\varphi' = \underbrace{F(b)\varphi(b)}_{=0} - \underbrace{F(a)\varphi(a)}_{=0} - \int_a^b F'\varphi$

so F' is the good g in the def. of $W^{1,1}$

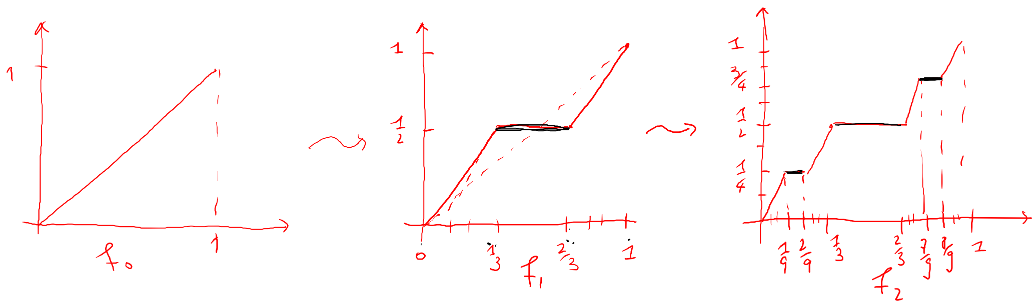
we will see that

Remark

¿ there exist functions which are continuous, of bounded variation, but which are not absolutely continuous?

continuous increasing   YES

Cantor function



$\rightsquigarrow f_3 \to f_4 \to \dots$

this is a Cauchy sequence in $\mathcal{C}([0,1])$ the limit is the Cantor function $\longrightarrow$ continuous and increasing.

it is differentiable on the sets in which it is constant

$$\left]\frac{1}{3},\frac{2}{3}\right[\cup \left]\frac{1}{9},\frac{2}{9}\right[\cup \left]\frac{7}{9},\frac{8}{9}\right[\cup \left]\frac{1}{27},\frac{2}{27}\right[\cup \left]\frac{7}{27},\frac{8}{27}\right[\cup \left]\frac{19}{27},\frac{20}{27}\right[\cup \left]\frac{25}{27},\frac{26}{27}\right[\cup \dots$$

compute the Lebesgue measure of this set

$$= \frac{1}{3}\cdot\sum_{n=0}^{+\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3} + \underbrace{\frac{1}{3}\cdot\left(\frac{2}{3}\right)}_{\frac{1}{9}+\frac{1}{9}} + \underbrace{\frac{1}{3}\cdot\left(\frac{2}{3}\right)^2}_{\frac{1}{27}+\frac{1}{27}+\frac{1}{27}+\frac{1}{27}} =$$

$$= 1$$

$$0 = \int_0^1 f'(t)\,dt < f(1) - f(0) = 1$$

Signed and complex measures

def. Let $(\Omega, \mathcal{F})$ be a measurable space $\left(\begin{array}{l}\Omega \text{ set} \\ \mathcal{F} \text{ is a } \sigma\text{-algebra of} \\ \text{subsets of } \Omega\end{array}\right)$

Let $\nu : \mathcal{F} \longrightarrow]-\infty, +\infty]$ (or $[-\infty, +\infty[$)

ν is called a signed measure if

i) $\nu(\phi) = 0$

ii) ν is countably additive

Remark countably additive in the case $\nu : \mathcal{F} \to]-\infty, +\infty]$ means the following:

take $(A_n)_n$ a sequence in $\mathcal{F}$ pairwise disjoint

let $A = \bigcup_n A_n \in \mathcal{F}$ $\quad (A_k \cap A_j = \phi \text{ if } k \neq j)$

- if $\nu(A) < +\infty$ then the series $\sum_n \nu(A_n)$ is absolutely convergent (i.e. $\sum_n |\nu(A_n)|$ is convergent)

  and $\sum_n \nu(A_n) = \nu(A)$

- if $\nu(A) = +\infty$ then

  denote $B_n = \begin{cases} A_n & \text{if } \nu(A_n) \geq 0 \\ \phi & \text{if } \nu(A_n) < 0 \end{cases}$, $\quad C_n = \begin{cases} \phi & \text{if } \nu(A_n) \geq 0 \\ A_n & \text{if } \nu(A_n) < 0 \end{cases}$

  we have $\sum_n \nu(B_n) = +\infty$, $\quad \sum_n -\nu(C_n) < +\infty$