

$$X \subseteq \mathbb{A}_k^n \quad \underline{I}(X) = \{ F \in k[x_1, \dots, x_n] \mid F(P) = 0 \ \forall P \in X \}$$

$$X = V(\alpha) \Rightarrow \alpha \subseteq \underline{I}(V(\alpha)) \quad \left\{ \begin{array}{l} P \in \mathbb{A}_k^n \quad P(a_1, \dots, a_n) \\ \underline{I}(P) = \{ F \mid F(P) = 0 \} \end{array} \right.$$

$$X \text{ closed:} \quad V(\underline{I}(X)) = X$$

$$Y \subseteq \mathbb{A}_k^n \text{ any set:} \quad V(\underline{I}(Y)) = \overline{Y} \quad \left\{ \begin{array}{l} P = V(x_1 - a_1, \dots, x_n - a_n) \end{array} \right.$$

$$\begin{cases} x_1 - a_1 = 0 \\ \vdots \\ x_n - a_n = 0 \end{cases}$$

$$\underline{I}(P) \supseteq \langle x_1 - a_1, \dots, x_n - a_n \rangle$$

↓

equality

To prove equality

1) we need to prove that $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is maximal,

2) to prove that if $F(a_1, \dots, a_n) = 0 \Rightarrow$
 $F = H_1(x_1 - a_1) + \dots + H_n(x_n - a_n)$

for suitable poly nomials $H_1, \dots, H_n$

If we prove $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is max. $\Rightarrow \underline{I}(P) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$

because $\underline{I}(P) \subsetneq k[x_1, \dots, x_n]$

How to prove that an ideal is maximal: $M \subseteq R$ any
 is maximal $\Leftrightarrow R/M$ is a field

$$\text{Im} \frac{K[x_1, \dots, x_n]}{\langle x_1 - a_1, \dots, x_n - a_n \rangle} \hookrightarrow$$

12

The polynomials $\langle x_1 - a_1, \dots, x_n - a_n \rangle$
 become zero

$$K[a_1, \dots, a_n] = K \text{ field}$$

$$\varphi_P: \begin{array}{ccc} K[x_1, \dots, x_n] & \longrightarrow & K \text{ evaluation at } P \\ \uparrow & & \uparrow \\ F & \longrightarrow & F(P) = F(a_1, \dots, a_n) \end{array}$$

$$K \cong \frac{K[x_1, \dots, x_n]}{I(P)}$$

$\Rightarrow I(P)$ maximal

φ_P is surjective; φ_P is a ring homomorphism

$$\text{Ker } \varphi_P = \{f \mid f(P) = 0\} = I(P)$$

$$x_1 - a_1, \dots, x_n - a_n \in \text{Ker } \varphi_P$$

$$\langle x_1 - a_1, \dots, x_n - a_n \rangle \subseteq I(P)$$

Taylor expansion of polynomials over any field

$$\underline{P(0_1, \dots, 0_n)}, \quad \underline{F \in \mathcal{I}(X)} \quad F = \underbrace{F(0_1, \dots, 0_n)} + \sum_{i=1}^n \frac{\partial F}{\partial x_i}(P)(x_i - 0_i) + \sum_{j=1}^n \frac{1}{2} \frac{\partial^2 F}{\partial x_i \partial x_j}(P)(x_i - 0_i)(x_j - 0_j) + \dots$$

$$= F(P) + G(x_1, \dots, x_n), \text{ where } G \in \langle x_1 - a_1, \dots, x_n - a_n \rangle$$

$$F(P) = 0 \iff F \in \langle x_1 - a_1, \dots, x_n - a_n \rangle$$

$\mathcal{I}(X) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$ and it is maximal

In $K[x_1, \dots, x_n]$, K any field, the ideals $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ are maximal.

$$\underline{\mathcal{I}(v(\alpha))}$$

§8. DERIVATIVES

8.1. Derivative of a polynomial. The derivative of a polynomial function, as of any function, is defined by limiting processes. The formula used for differentiating such functions, however, involves no such processes in any explicit fashion. We shall now see that a purely formal application of this formula will give us many of the properties of derivatives.

For any element

$$f = a_0 + a_1x + \cdots + a_nx^n \quad \leftarrow$$

of $D[x]$ we define the *derivative* of f with respect to x to be

$$f' = a_1 + 2a_2x + \cdots + na_nx^{n-1}. \quad \leftarrow$$

The (partial) derivative of $f(x_1, \dots, x_r)$ with respect to x_i may then be defined by regarding $f(x_1, \dots, x_r)$ as a polynomial over $D[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r]$. The following properties of derivatives will be useful:

- (i) $(f + g)' = f' + g'$.
- (ii) If a is a constant, $a' = 0$ and $(af)' = af'$.
- (iii) $(fg)' = f'g + fg'$; $(f^n)' = nf^{n-1}f'$.
- (iv) $f(g_1(x), \dots, g_r(x))' = \sum_i f_i(g_1(x), \dots, g_r(x))g'_i(x)$,

where $f_i(x_1, \dots, x_r)$ is the derivative of f with respect to x_i , $i = 1, \dots, r$.

(i) and (ii) follow at once from the definition, and the first part of (iii) can be easily checked; the last part of (iii) is obtained by repeated application of the first part. (iv) may be proved first in the case in which f consists of a single term, by use of (iii), and then in the general case by use of (i) and (ii).

Derivatives have certain undesirable properties for polynomials over a domain of characteristic $p \neq 0$. This is due to the fact that for such a domain any polynomial of the type $a_0 + a_1x^p + a_2x^{2p} + \cdots$ has zero for its derivative. Hence throughout the rest of this section we shall assume that D has characteristic zero. We also assume that D has unique factorization.

Our most important use of derivatives will be in the investigation of multiple factors of polynomials. A fundamental theorem in this connection is the following.

THEOREM 8.1. *If g is irreducible, and not a constant, $g^2 \mid f$ if and only if $g \mid f$ and $g \mid f'$.*

PROOF. If $g \mid f$ then $f = gh$, and so $f' = gh' + g'h$. If, also, $g \mid f'$, then $g \mid g'h$. Since $\deg g' < \deg g$, $g \nmid g'$, and so $g \mid h$, since g is irreducible. Therefore, $g^2 \mid f$. Conversely, if $f = g^2k$ then $f' = 2gg'k + g^2k'$, and so $g \mid f$ and $g \mid f'$.

$F(t) = f(a_1 + (x_1 - a_1)t, \dots, a_r + (x_r - a_r)t)$ in powers of t . Using (iv) we obtain

$$F(t) = f(a_1, \dots, a_r) + \sum_i f_i(x_i - a_i)t + \frac{1}{2!} \sum_{i,j} f_{ij}(x_i - a_i)(x_j - a_j)t^2 + \dots$$

The desired result is obtained by putting $t = 1$.

8.3. Exercises. 1. Properties (i), ..., (iv) will hold for rational functions if and only if we define $(f/g)'$ to be $(f'g - fg')/g^2$.

2. Extend Theorems 8.1 and 8.2 from squares to n th powers.

3. Prove Theorem 8.2 and its generalization for domains of any characteristic.