MECCANICA RAZIONALE

Ing Civile & Ambientale

Navale

17 marzo 2021

Campi di forza, forze conservative

Come formulazione Statica & Dinamica

del corpo rigido

- gradi di libertà → coordinate libere

- cinematica : $\frac{d}{dt} \underline{x}_P(t) = \frac{d}{dt} \underline{x}_O(t) + \underline{\omega} \wedge (\underline{x}_P - \underline{x}_O)$

$$\underline{v}_P = \underline{v}_O + \underline{\omega} \wedge (\underline{x}_P - \underline{x}_O)$$

↳ atto di moto

- L.V. (spostamenti virtuali)

$$LV = \sum_{B \in \mathcal{A}} \underline{F}_B^a \cdot d\underline{x}_B = \sum_{i=1}^{\ell} Q_i \, \delta q_i = 0$$

$$Q_i = 0$$

$$\left(\sum \underline{F}_B \cdot \frac{\partial x_B}{\partial q_i} \right)$$

LV rigido $\longrightarrow$ $\underline{R}$ risultante forze

$\underline{M}(O)$ momento risultante

- $\underline{F}^a$ forze attive

$\rightarrow$ forze posizionali

$\Rightarrow$ forze conservative : energia meccanica totale (cinetica + potenziale) si conserva

Conservativa : il lavoro svolto per spostare

non dipende dalla traiettoria

$\longrightarrow$ $\exists$ funzione $V(x)$ tale che il lavoro $\quad -\left(V(\underline{x}_1) - V(\underline{x}_0) \right)$

$\underline{x}_E$ config. di equilibrio

$\underline{x}_E + \delta \underline{x}$

perché $\underline{x}_E$ equilibrio

$$LV = -\left[V(\underline{x}_E + \delta \underline{x}) - V(\underline{x}_E) \right] = 0$$

$$= -\nabla V\Big|_{\underline{x}_E} \cdot \delta \underline{x} = -dV$$

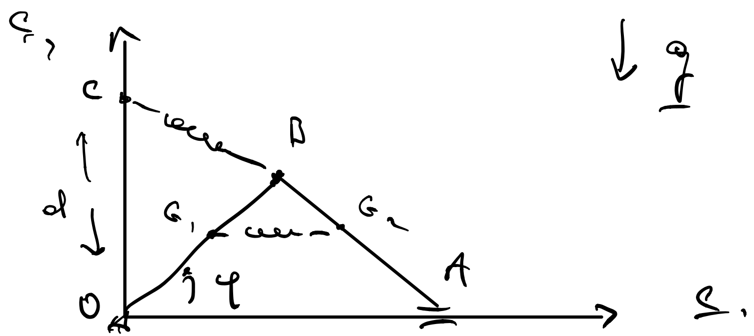
regolarità

$\underline{x}_E$ equilibrio $\longleftrightarrow$ $\underline{x}_E$ punto di stazionarietà

Equilibrio dei sistemi olonomi

a l gradi di libertà e soggetti

a forze conservative



$\overline{OB} = \overline{AB} = l$

m M

c

$$V = \underset{\text{peso}}{V_1} + \underset{\text{molla esterna}}{V_2} + \underset{\text{molla interna}}{V_3}$$

$$LV_{peso} = m \underline{g} \cdot \delta \underline{x}_{G_1} + M \underline{g} \cdot \delta \underline{x}_{G_2} = - dV_1$$

$$LV_{molla_1} = -c \, (\underline{x}_B - \underline{x}_C) \cdot \delta \underline{x}_B = - dV_2$$

$$LV_{molla_2} = -c \, (\underline{x}_{G_1} - \underline{x}_{G_2}) \cdot \delta (\underline{x}_{G_1} - \underline{x}_{G_2})$$

$$= - dV_3$$

$$V = m g y_{G_1} + m g y_{G_2} + \frac{c}{2} \left[x_B^2 + (y_B - d)^2 \right] + \frac{c}{2} (x_{G_1} - x_{G_2})^2$$

$$= (m + M) g \frac{L}{2} \underline{\sin\varphi} + \frac{c}{2} \left[L^2 \cos^2\varphi + \underbrace{(L \sin\varphi - d)^2}_{L^2 \sin^2\varphi + d^2 - 2L \sin\varphi \, d} \right] + \underline{\frac{c}{2} L^2 \cos^2\varphi}$$

$$= \left[(m + M) g \frac{L}{2} - c L d \right] \underline{\sin\varphi} + \frac{c}{2} L^2 \underline{\cos^2\varphi}$$

$$+ \underbrace{\frac{c}{2} \left[L^2 \cos^2\varphi + L^2 \sin^2\varphi + d^2 \right]}_{\cos^2\varphi + \sin^2\varphi = 1}$$

$$= V(\varphi)$$

configurazioni di equilibrio

$\to$ stazionarietà di $V = V(\varphi)$

minimi $\longrightarrow$ equilibrio stabile

$$\frac{d}{d\varphi} V(\varphi) = \left[(m+M) g \frac{L}{2} - c L d \right] \cos\varphi +$$

$$- c L^2 \sin\varphi \cos\varphi$$

$$= c L^2 \left\{ \frac{1}{cL} \left[(m+M) \frac{g}{2} - c d \right] - \sin\varphi \right\} \cos\varphi$$

$$= c L^2 \left(\gamma - \sin\varphi \right) \cos\varphi$$

$$\gamma := \frac{1}{cL} \left[(m+M) \frac{g}{2} - c d \right]$$

Assumiamo $-\pi < \varphi \leq \pi$ $\leftarrow$

1) $|\gamma| > 1 \implies (\gamma - \sin\varphi) \neq 0 \quad \forall \varphi$

$$\frac{dV}{d\varphi} = 0 \qquad \cos\varphi = 0 \quad , \quad \varphi = \frac{\pi}{2}, \ \varphi = -\frac{\pi}{2}$$

$\to$ due configurazioni di equilibrio

2) $|\gamma| < 1 \Rightarrow \frac{dU}{d\varphi} = 0$

$\cos\varphi = 0$ oppure $\sin\varphi = \gamma$

Allora 4 configurazioni di equilibrio

$\varphi = \frac{\pi}{2}$, $\varphi = -\frac{\pi}{2}$, $\varphi = \varphi_1$, $\varphi = \varphi_2$

$\sin\varphi_1 = \gamma$ $\quad \sin\varphi_2 = \gamma$

Stabilità di queste configurazioni:

$$\frac{d^2V}{d\varphi^2} = \frac{d}{d\varphi}\left\{cL^2(\gamma - \sin\varphi)\cos\varphi\right\} =$$

$$= cL^2\left[-\sin\varphi\,(\gamma - \sin\varphi) - \cos^2\varphi\right]$$

1) $|\gamma| > 1 \Rightarrow$ eq per $\varphi = \pm\frac{\pi}{2}$

$$V''\left(\frac{\pi}{2}\right) = -cL^2(\gamma - 1) = cL^2(1-\gamma)$$

$$V''\left(-\frac{\pi}{2}\right) = cL^2(1+\gamma)$$

$\gamma > 1$ $\begin{cases} V''\left(\frac{\pi}{2}\right) < 0 \longrightarrow \text{instabile} \\ V''\left(-\frac{\pi}{2}\right) > 0 \longrightarrow \text{stabile} \end{cases}$

$$\gamma < -1 \quad \begin{cases} V''\left(\frac{\pi}{2}\right) > 0 & \to \text{stabile} \\ V''\left(-\frac{\pi}{2}\right) < 0 & \to \text{instabile} \end{cases}$$

2) Se $|\gamma| < 1 \implies \varphi = \pm\frac{\pi}{2}, \varphi_1, \varphi_2$

$$V''\left(\frac{\pi}{2}\right) = cL^2(1-\gamma) > 0 \quad \to \text{stabile}$$

$$V''\left(-\frac{\pi}{2}\right) = cL^2(1+\gamma) > 0 \quad \to \text{stabile}$$

$$V''(\varphi_1) = cL^2\left[-\sin\varphi_1 \underbrace{(\gamma - \sin\varphi_1)}_{=0} - \cos^2\varphi_1\right]$$

$$= -cL^2\cos^2\varphi_1 < 0 \quad \text{instabile}$$

$$V''(\varphi_2) = -cL^2\cos^2\varphi_2 < 0 \quad \text{instabile}$$

Allora

$\gamma < -1$	$-1 < \gamma < 1$	$\gamma > 1$
$\frac{\pi}{2}$ stabile	$\frac{\pi}{2}$ stabile	$\frac{\pi}{2}$ instabile
$-\frac{\pi}{2}$ instabile	$-\frac{\pi}{2}$ stabile	$-\frac{\pi}{2}$ stabile
	φ_1 instabile	
	φ_2 instabile	

$$f = \frac{1}{cL}\left[\underbrace{(m+M)\frac{g}{2}}_{\text{peso}} - \underbrace{cd}_{\text{elastica}}\right]$$

Configurazioni di equilibrio possono cambiare carattere (ad esempio passare da stabili a instabili)

Seconda parte

Se un solo grado di libertà

$V = V(q)$, $LV = -dV = -\frac{dV}{dq}\,\delta q$

$= Q\,\delta q$

$$\Rightarrow \boxed{Q = -\frac{dV}{dq}}$$

All'equilibrio $Q = 0 \longleftrightarrow \frac{dV}{dq} = 0$

Supponiamo di avere l gradi di libertà: $\underline{q} = (q_1, \dots, q_l)$

$V = V(q_1, \dots, q_l)$ (regolare)

$$L.V. = -dV = -\left[\frac{\partial V}{\partial q_1}\delta q_1 + \dots + \frac{\partial V}{\partial q_l}\delta q_l\right]$$

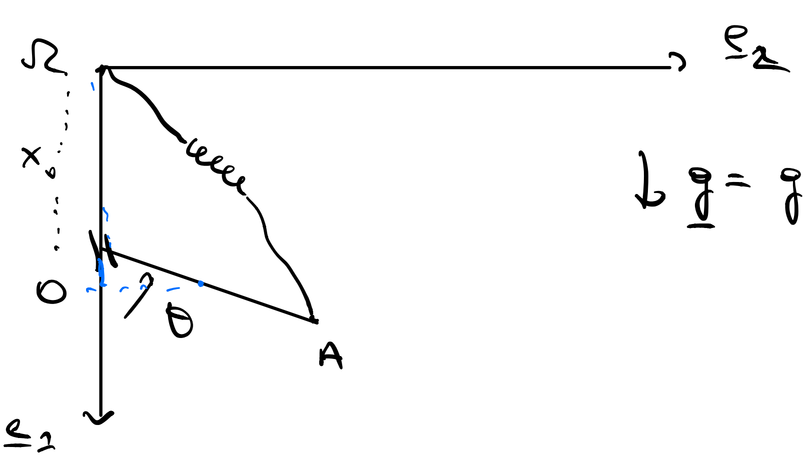
$$= Q_1 \delta q_1 + \dots + Q_l \delta q_l$$

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad \forall\, i = 1, \dots, l$$

All'equilibrio: $\dfrac{\partial V}{\partial q_i} = 0 \qquad i = 1, \dots, l$

$$V = V(q_1, \dots, q_l) \longrightarrow \frac{\partial V}{\partial q_i} = 0$$

Esempio



$\underline{g} = g\underline{e}_1$

$\bar{OA}$ occupa piano verticale

due gradi di libertà (x_0, θ)

$$V = -m\underline{g}\cdot\underline{x}_G + \frac{c}{2}\|\underline{x}_A\|^2$$

$$= -mg\left(\frac{L}{2}\cos\theta + x_0\right) +$$

$$+\frac{c}{2}\left[\underbrace{(x_0 + L\cos\theta)^2}_{x_0^2 + L^2\cos^2\theta + 2x_0 L\cos\theta} + L^2\sin^2\theta\right]$$

$$= -mg\frac{L}{2}\cos\theta - mg\,x_0 + \frac{c}{2}(x_0^2 + 2Lx_0\cos\theta)$$

$$+\frac{c}{2}L^2$$

Forze generalizzate

$$-Q_{x_0} = \frac{\partial V}{\partial x_0} = -\underline{R}^a\cdot\underline{e}_1 = -mg + cx_0 + cL\cos\theta$$

$$-Q_\theta = \frac{\partial V}{\partial \theta} = -\underline{M}^a(O)\cdot\underline{e}_3 = \left(\frac{mg}{2} - cx_0\right)L\sin\theta$$

Configurazioni di equilibrio

$$\begin{cases} Q_{x_0} = 0 \\ Q_\theta = 0 \end{cases} \Rightarrow \begin{cases} -mg + cx_0 + cL\cos\theta = 0 \\ \left(\frac{mg}{2} - cx_0\right)L\sin\theta = 0 \end{cases}$$

Prendiamo la seconda equazione:

$$\sin\theta = 0 \qquad \text{oppure} \qquad x_0 = \frac{mg}{2c}$$

Sostituiamo nella prima $\quad (-\pi < \theta \leq \pi)$

1) se prendiamo $\sin\theta = 0 \rightarrow \begin{cases} \theta = 0 \\ \theta = \pi \end{cases}$

$$\begin{cases} \cos\theta = 1 \\ \cos\theta = -1 \end{cases} \rightarrow \begin{cases} x_0 = \frac{mg}{c} - L \\ x_0 = \frac{mg}{c} + L \end{cases}$$

2) se prendiamo $x_0 = \frac{mg}{2c}$ e sostituiamo nella prima equazione

$$-mg + c x_0 + cL\cos\theta = 0$$

$$\uparrow \; x_0 = \frac{mg}{2c}$$

$$\rightarrow \cos\theta = \frac{mg}{2cL}$$

da θ se $\frac{mg}{2cL} < 1$

Quindi:

se $\frac{mg}{2cL} > 1 \Rightarrow$ 2 configurazioni di equilibrio

$$q_1 = \left(x_0 = \frac{mg}{c} - L , \; \theta = 0 \right)$$

$$q_2 = \left(x_0 = \frac{mg}{c} + L , \; \theta = \pi \right)$$

se $\frac{mg}{2cL} < 1 \Rightarrow$ 4 configurazioni di equilibrio

q_1, q_2 come sopra

$$q_3 = \left(x_0 = \frac{mg}{2c}, \ \theta_1 \ \text{t.c.} \ \cos\theta_1 = \frac{mg}{2cL} \right)$$

$$q_4 = \left(x_0 = \frac{mg}{2c}, \ \theta_2 \ \text{t.c.} \ \cos\theta_2 = \frac{mg}{2cL} \right)$$

$(\theta_2 = -\theta_1)$

Vogliamo capire se l'equilibrio è stabile o meno.

$$q \rightarrow \frac{dV}{dq} = 0 \qquad \frac{d^2V}{dq^2} : > 0, < 0$$

$$q_1, q_2 \rightarrow \frac{\partial V}{\partial q_1} = 0 \quad \frac{\partial V}{\partial q_2} = 0, \quad \text{Hess } V$$

$$\frac{\partial V}{\partial x_0} = -mg + cx_0 + cL\cos\theta$$

$$\frac{\partial V}{\partial \theta} = \left(\frac{mg}{2} - cx_0\right) L\sin\theta$$

$$\text{Hess } V = \begin{pmatrix} \frac{\partial^2 V}{\partial x_0^2} & \frac{\partial^2 V}{\partial x_0 \partial \theta} \\ \frac{\partial^2 V}{\partial \theta \partial x_0} & \frac{\partial^2 V}{\partial \theta^2} \end{pmatrix}$$

$$= \begin{pmatrix} c & -cL\sin\theta \\ -cL\sin\theta & \left(\frac{mg}{2} - cx_0\right) L\cos\theta \end{pmatrix}$$

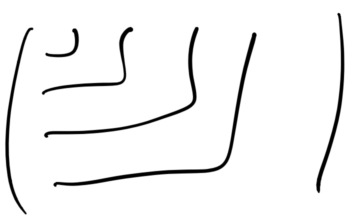
dall'analisi:

Hess V definita positiva → minimo locale

Hess V definita negativa → massimo locale

Hess V indefinita → punto sella

Criterio di Sylvester: matrice simmetrica è definita positiva se e solo se tutti i suoi minori principali sono positivi



$$: \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix}$$

$$A_{11} > 0$$

$$\det = A_{11} A_{22} - A_{12}^2 > 0$$

Nel nostro caso

$$\begin{pmatrix} c & -cL\sin\theta \\ -cL\sin\theta & \left(\frac{mg}{2} - c x_0\right) L\cos\theta \end{pmatrix}$$

$A_{11} = c > 0$

$$\det \mathrm{Hess}\, V = c\left(\frac{mg}{2} - c x_0\right) L\cos\theta - c^2 L^2 \sin^2\theta$$

(calcolarlo nelle configurazioni di equilibrio)

$$\underline{q}_1 = \left(x_0 = \frac{mg}{c} - L,\ \theta = 0\right)$$

$$\det \mathrm{Hess} = cL\left(cL - \frac{mg}{2}\right) > 0$$

se $\frac{mg}{2cL} < 1$

$\Rightarrow$ $\underline{q}_1$ è stabile solo se $\frac{mg}{2cL} < 1$

$$\underline{q}_2 = \left(x_0 = \frac{mg}{c} + L,\ \theta = \pi\right)$$

$$\det \mathrm{Hess}\, V = cL\left(cL + \frac{mg}{2}\right) > 0$$

sempre

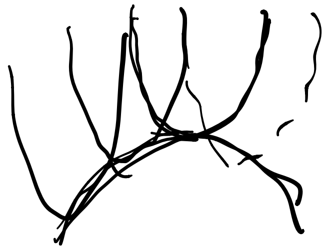
$\Rightarrow$ $\underline{q}_2$ sempre stabile

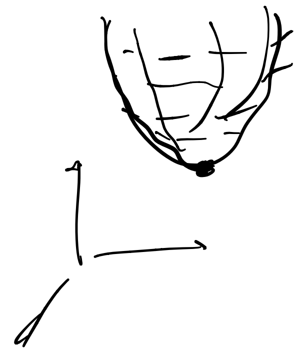
$$\underline{q}_3, \underline{q}_4 : \left(x_0 = \frac{mg}{2c},\ \cos\theta_{1,2} = \frac{mg}{2cL}\right)$$

$$\det \mathrm{Hess}\, V = -c^2 L^2 \sin^2\theta_{1,2} < 0$$

$\underline{q}_3, \underline{q}_4$ sempre instabili

Terza parte





Consideriamo un sistema dotato vincoli fissi, l gradi di libertà

$$(q_1, \dots, q_l)$$

Supponiamo che il sistema evolva lungo $\underline{q}(t) = (q_1(t), \dots, q_l(t))$

$$\to \underline{x}_A(t) = \underline{x}_A(q_1(t), \dots, q_l(t))$$

$$\underline{v}_A = \frac{d}{dt}\underline{x}_A(t) = \frac{\partial}{\partial q_1}\underline{x}_A \, \dot{q}_1 + \dots + \frac{\partial \underline{x}_A}{\partial q_l}\dot{q}_l$$

. (dalla fisica $L = \underline{F} \cdot \underline{x}$

$$W = \underline{F} \cdot \underline{v} \to L = \int W(t)\,dt \;)$$

$$W(t) = \underline{F}_A \cdot \underline{v}_A = \underline{F}_A \cdot \left(\frac{\partial \underline{x}_A}{\partial q_1}\dot{q}_1\right) + \dots + \left(\frac{\partial \underline{x}_A}{\partial q_l}\dot{q}_l\right)$$

$$= Q_1 \dot{q}_1 + \dots + Q_\ell \dot{q}_\ell$$

"forma differenziale"

$$\text{Lavoro} = \int_{\tau_0}^{\tau_1} W(\tau)\, d\tau =$$

$$= \int_{\tau_0}^{\tau_1} \left(Q_1 \dot{q}_1 + \dots + Q_\ell \dot{q}_\ell \right) d\tau$$

Il lavoro dipende solo dalle due configurazioni τ_0 e τ_1 $\Longleftrightarrow$ $\exists$ una funzione $V(q_1, \dots, q_\ell)$ tale che

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad i = 1, \dots, \ell$$

(la forma differenziale si dice "esatta")

Condizione necessaria perché

$$Q_i = -\frac{\partial V}{\partial q_i}$$

è che

$$\boxed{\frac{\partial}{\partial q_j} Q_i = \frac{\partial}{\partial q_i} Q_j} \qquad i,j = 1,\dots,\ell$$

condizioni di compatibilità (forma differenziale "chiusa")

Condizione necessaria perché esista

V

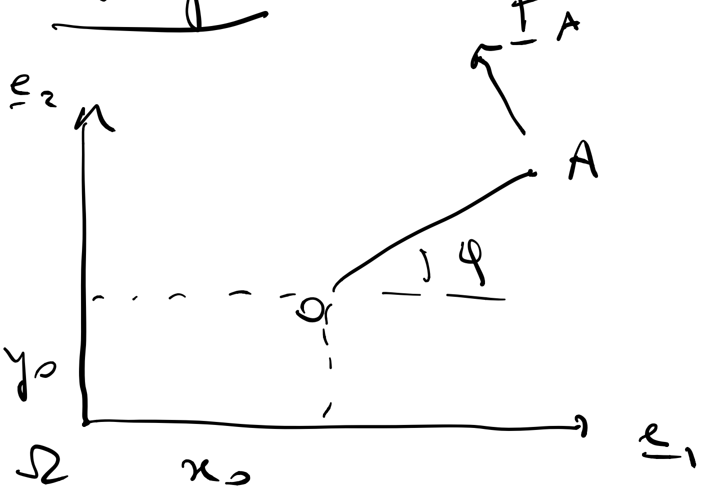
Dim se $Q_i = -\dfrac{\partial V}{\partial q_i}$ $\quad i = 1,\dots,\ell$

$$\frac{\partial}{\partial q_j} Q_i = \frac{\partial}{\partial q_j}\left(-\frac{\partial V}{\partial q_i}\right) =$$

$$= \frac{\partial}{\partial q_i}\left(-\frac{\partial V}{\partial q_j}\right) = \frac{\partial}{\partial q_i} Q_j$$

condizione necessaria ma non sufficiente

Esempio



$$\underline{F}_A = -F\sin\varphi\,\underline{e}_1 + F\cos\varphi\,\underline{e}_2$$

$$\overline{OA} = L$$

$$\underline{M}(O) = FL\,\underline{e}_3$$

Vediamo subito:

$$Q_{x_0} = -F\sin\varphi, \qquad Q_{y_0} = F\cos\varphi$$

$(\underline{R}\cdot\underline{e}_1)$ $\qquad$ $(\underline{R}\cdot\underline{e}_2)$

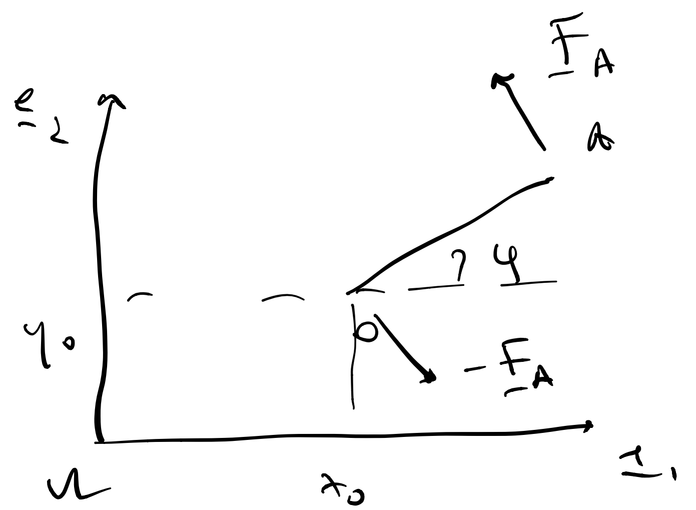
$$Q_\varphi = FL$$

$$\frac{\partial}{\partial\varphi} Q_{x_0} = \frac{\partial}{\partial\varphi}\left(-F\sin\varphi\right) = -F\cos\varphi$$

$$\neq \qquad \stackrel{?}{=} \frac{\partial}{\partial x_0} Q_\varphi = \frac{\partial}{\partial x_0} FL = 0$$

$\longrightarrow$ forza non è conservativa

Nota: può capitare che anche se le singole forze non sono conservative, LV totale sia $-dV$



$$Q_{x_0} = \underline{R}\cdot\underline{e}_1 = 0$$

$$Q_{y_0} = \underline{R}\cdot\underline{e}_2 = 0$$

$$(\underline{R} = \underline{F}_A - \underline{F}_A = 0)$$

$$Q_\varphi = \underline{M}(O)\cdot\underline{e}_3 = \\ = FL$$

$$LV = F L \, \delta\varphi = \delta(\underbrace{F L \varphi}_{-V})$$

$$(\underline{M}(O) \cdot \underline{e}_3)\, \delta\varphi$$

<u>Caso importante</u> : sistemi ad un solo grado di libertà soggetti solo a forze posizionali

$$LV = Q(q)\, \delta q = d(-V)$$

$$V = -\int Q \quad \text{primitiva di } Q$$

vale solo per 1 coordinata libera

Torniamo ad un esempio precedente

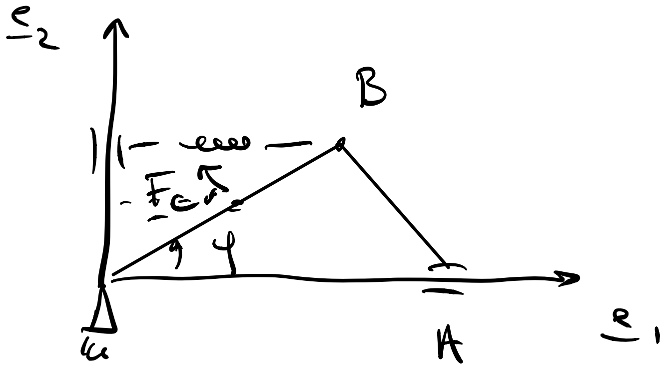
$\exists\, V$ tale che
$$\frac{\partial V}{\partial x_0} = -Q_{x_0} = F \sin\varphi$$

$$\frac{\partial V}{\partial y_0} = -Q_{y_0} = -F \cos\varphi$$

$$\frac{\partial V}{\partial \varphi} = -Q_\varphi = -FL \quad ?$$

no, non esiste

Esempio



$OB = BA = L$

$OC = \frac{L}{2}$

$F_C = F \cos\varphi \, e_2 - F \sin\varphi \, e_1$

$$U_{molla} = \frac{c}{2} x_B^2 = \frac{c}{2} (L \cos\varphi)^2$$

$$F_c : \quad LV = F_c \cdot \delta x_c =$$

$$= (F \cos\varphi \, e_2 - F \sin\varphi \, e_1) \cdot \left(\frac{L}{2} (-\sin\varphi) \delta\varphi \, e_1 + \frac{L}{2} \cos\varphi \, \delta\varphi \, e_2 \right)$$

$$= \frac{FL}{2} (\sin^2\varphi + \cos^2\varphi) \, \delta\varphi = \frac{FL}{2} \delta\varphi$$

$$= -\frac{dV}{d\varphi} \delta\varphi \quad \text{dove} \quad V = -\frac{FL}{2} \varphi$$

Quindi anche se F non conservativa

$$V(\varphi) = -\frac{FL}{2} \varphi + \frac{c}{2} L^2 \cos^2\varphi$$

condizione di equilibrio

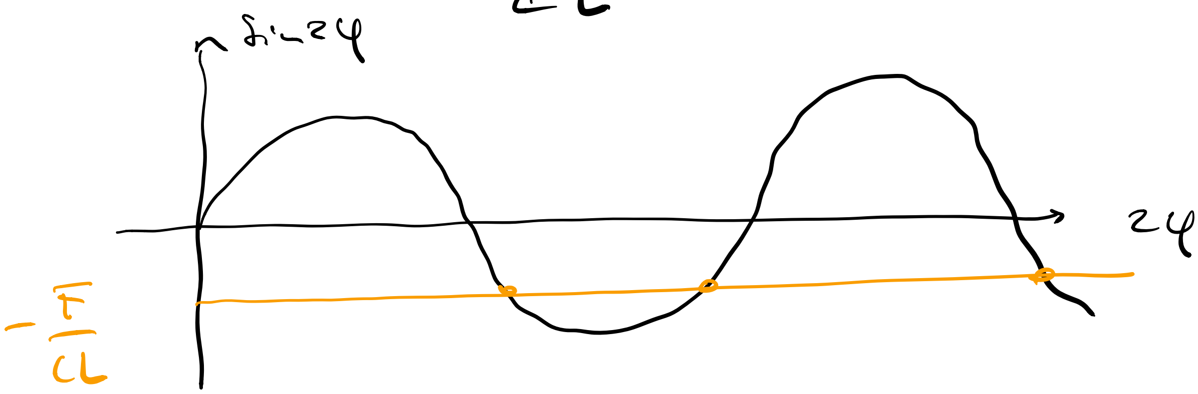
$$-Q = \frac{dV}{d\varphi} = -\frac{L}{2}\left(F + cL \underbrace{2\cos\varphi\sin\varphi}_{\sin 2\varphi}\right)$$

$= 0$

equilibrio per $\sin 2\varphi = -\frac{F}{cL}$

$\exists$ conf di equilibrio solo per

$$0 \leq \frac{F}{cL} < 1$$



$$V''(\varphi) = -c\left(L^2\cos 2\varphi\right)$$