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Hilbert Nullstellensatz K algebraically closed

$\alpha \subset K[x_1, \dots, x_n]$, $\quad I(V(\alpha)) = \sqrt{\alpha}$.

It will follow from Normalization Lemma:

A K-algebra, K any field

A integral domain and finitely generated K-algebra

$\text{tr.d.}\, Q(A)/K = r \implies \exists\ z_1, \dots, z_r \in A$ algebraically

indep. over K s.t. A is integral over $K[z_1, \dots, z_r]$.

$A = K[y_1, \dots, y_n]$, $\quad Q(A) = K(y_1, \dots, y_n)$: $\quad r \le n$

$K[z_1, \dots, z_r]$ is isomorph. to the polyn. ring

$K \subseteq K[z_1, \dots, z_r] \subseteq A$

Finitely generated K-algebras: if $A = K[y_1, \dots, y_n]$

$$\varphi : K[x_1, \dots, x_n] \longrightarrow A = K[y_1, \dots, y_n] \quad \text{surjective ring homom.}$$

$$F \longrightarrow F(y_1, \dots, y_n)$$

$$\implies A \simeq K[x_1, \dots, x_n] / \operatorname{Ker} \varphi$$

$\operatorname{Ker} \varphi = \{ F \mid F(y_1, \dots, y_n) = 0 \}$ algebraic dependence

relations among the generators of A

$$A = \frac{K[x_1, \dots, x_n]}{\alpha} \qquad K \overset{j}{\hookrightarrow} K[x_1, \dots, x_n] \xrightarrow{p} \frac{K[x_1 \dots x_n]}{\alpha} = A$$

$p \circ j$ is injective because the kernel is (0).

$\Longrightarrow A$ is a K-algebra ; $[x_i] = \xi_i \in A$

$$[F(x_1, \dots, x_n)] = F(\xi_1, \dots, \xi_n)$$

$$A = K[\xi_1, \dots, \xi_n]$$

A is a finitely gen. K-algebra $\Longleftrightarrow$ A is a quotient of a polynomial ring in finitely many variables.

Proof of the Nullstellensatz

1) K alg. closed, $K[x_1, \dots, x_n] \supsetneq \mathfrak{m}$ maximal ideal

$\Longrightarrow \exists\, a_1, \dots, a_n \in K$ s.t. $\mathfrak{m} = \underline{\langle x_1 - a_1, \dots, x_n - a_n \rangle}$.

$K' := \dfrac{K[x_1, \dots, x_n]}{\mathfrak{m}}$ is a field and it is a finitely generated K-algebra by $\xi_1, \dots, \xi_n$: integral domain $\Longrightarrow$ we can apply norm.

lemma

$Q(K') = K'$; $\boxed{r = \operatorname{trd} K'/K} = 0$ $\Longrightarrow \exists\, z_1, \dots, z_r \in K'$

s.t. K' is integral over $\underline{K[z_1, \dots, z_r]}$.

We claim that also $K[z_1, \dots, z_r]$ is a field.

$\underset{\neq 0}{f} \in K[z_1, \dots, z_r] \subseteq K'$ field: f is invertible in K'

$\exists\, f^{-1} \in K'$: we want to prove that $f^{-1} \in K[z_1, \dots, z_r]$

f^{-1} is integral over $K[z_1, \dots, z_r]$: $\exists\, b_1, \dots, b_s \in K[z_1, \dots, z_r]$ s.t.

$$(f^{-1})^s + b_1 (f^{-1})^{s-1} + \dots + b_s = 0 \quad / (f)^{s-1}$$

$$f^{-1} + b_1 + b_2 f + \dots + b_s f^{s-1} = 0$$

$$f^{-1} = -(b_1 + b_2 f \dots) \in K[z_1, \dots, z_r]$$

$\Longrightarrow K[z_1, \dots, z_r]$ is a field.

$K[z_1, \dots, z_r]$ is a polyn. ring in r variables
field } $\Rightarrow r = 0$

A variable is never invertible

$\Rightarrow K'$ is an algebraic extension of K,

K is algebraically closed $\Longrightarrow$ $\boxed{K' \simeq K}$

$$K \xrightarrow{j} K[x_1, \dots, x_n] \xrightarrow{p} K[x_1, \dots, x_n] / \mathfrak{M} = \underline{K' \xrightarrow[\varphi]{\sim} K}$$

$$x_i \longrightarrow [x_i] \longrightarrow \boxed{a_i}$$

$$x_i - a_i \longrightarrow [x_i] - [a_i] \longrightarrow a_i - a_i = 0$$

$p \circ j$ is injective : $[a_i]$ contains only one constant

$$x_i - a_i \in \operatorname{Ker}(\varphi \circ p) = \mathfrak{M} = \operatorname{Ker} p$$

$$x_1 - a_1, \dots, x_n - a_n \in \mathfrak{M}$$

$$\langle x_1 - a_1, \dots, x_n - a_n \rangle \leq \mathfrak{M} \quad \text{maximal}$$

maximal $\Longrightarrow$ equality

$$\mathfrak{M} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$$

WEAK FORM OF NULLSTELL.

2) $\mathfrak{a} \subsetneq K[x_1 \dots x_n]$, K alg. closed $\Longrightarrow$

$V(\mathfrak{a}) \subseteq \mathbb{A}^n_K$ is non-empty i.e $\mathfrak{a}$ has at least one zero in $\mathbb{A}^n_K$.

$\mathfrak{a} \subseteq \mathfrak{m}$ maximal ideal

$V(\mathfrak{a}) \supseteq V(\mathfrak{m}) = V(x_1 - a_1, \dots, x_n - a_n) = \{ \underline{(a_1, \dots, a_n)} \}$

$\Rightarrow$ $V(\mathfrak{a})$ is non-empty.

Maximal ideals containing $\mathfrak{a}$ give points of $V(\mathfrak{a})$

$\mathfrak{a}$ proper $\Rightarrow V(\mathfrak{a}) \neq \emptyset$

3) K alg. closed, $\mathfrak{a} \subset K[x_1 \dots x_n]$: $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Known: $\sqrt{\mathfrak{a}} \subseteq I(V(\mathfrak{a}))$ always true

To prove: $I(V(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$ i.e. if F vanishes in all points of $V(\mathfrak{a})$, then $\exists r > 0$ s.t. $F^r \in \mathfrak{a}$

If $F = 0$, it belongs to all ideals. .

$$1 = H_1(x_1 \dots x_n, t)\, G_1 + \dots + H_s(x_1 \dots x_n, t)\, G_s +$$

$$+ H_{s+1}(x_1 \dots x_n, t)\left(\underline{\underline{tF(x_1 \dots x_n) - 1}}\right)$$ equality of polynomials

Idea: to specialize the variables and precisely replace t with $\frac{1}{F}$; we can do it in $K(x_1, \dots, x_n) \ni \frac{1}{F}$

$$\varphi : K[x_1 \dots x_n, t] \longrightarrow K(x_1, \dots, x_n)$$ ring homom.

$$G(x_1 \dots x_n, t) \longmapsto G\left(x_1 \dots x_n, \frac{1}{F}\right)$$

Apply φ: $$1 = H_1\left(x_1 \dots x_n, \frac{1}{F}\right) G_1(x_1 \dots x_n) +$$

$$+ \dots + H_s\left(x_1 \dots x_n, \frac{1}{F}\right) G_s(x_1 \dots x_n) + \cancel{H_{s+1}(\dots)\Big(\underbrace{\tfrac{1}{F} F - 1}_{0}\Big)}$$

$$H_1\left(x_1 \dots x_n, \frac{1}{F}\right) = \frac{A_1'(x_1 \dots x_n)}{F^{r_1}}$$

$$1 = \frac{H_1'(x_1 \dots x_n)}{F^{r_1}} G_1(x_1 \dots x_n) + \dots + \frac{H_s'(x_1 \dots x_n)}{F^{r_s}} G_s(x_1 \dots x_n) \Big/ F^{r}$$

$$F^{r} = H_1'' G_1 + \dots + H_s'' G_s \in \alpha$$

$$\Rightarrow \quad F \in \sqrt{\alpha} .$$

<u>Corollaries</u> K algebraically closed

1) There is bijection between

$$\left\{ \begin{matrix} \text{affine varieties in } \mathbb{A}^n_K \\ \text{,, Zariski closed} \end{matrix} \right\} \xleftrightarrow[V]{I} \left\{ \begin{matrix} \text{radical ideals} \\ \text{in } K[x_1 \dots x_n] \end{matrix} \right\}$$

$$\alpha = \sqrt{\alpha}$$

$X \subset \mathbb{A}^n$ Zariski closed : $X = V(I(X))$

$\alpha = \sqrt{\alpha} \subset K[x_1 \dots x_n]$ $\quad \alpha = \sqrt{\alpha} = I(V(\alpha))$

On these two sets $V \circ I = id$, $I \circ V = id$

This bijection reverses inclusions.

2) $V(\alpha) \cap V(\beta) = V(\alpha + \beta)$

- $V(\alpha) \cup V(\beta) = V(\alpha \cap \beta) = V(\alpha\beta)$

$X, Y \subseteq \mathbb{A}^n_K$, K alg. closed, X, Y zariski closed

$I(X \cup Y)$, $I(X \cap Y)$?

$I(X \cap Y) = I(V(I(X)) \cap V(I(Y))) =$

$= I(V(I(X) + I(Y))) \underset{NS}{=} \sqrt{I(X) + I(Y)}$

$I(X \cup Y) = I(V(I(X) \cap I(Y))) = I(V(I(X)I(Y)))$

$= \sqrt{I(X) \cap I(Y)} = \sqrt{I(X)I(Y)} = \underline{\underline{I(X) \cap I(Y)}}$

$\forall \alpha, \beta \subseteq K[x_1 \dots x_n]$, $\sqrt{\alpha \cap \beta} = \sqrt{\alpha\beta} = \sqrt{\alpha} \cap \sqrt{\beta}$

$\sqrt{I(X) \cap I(Y)} = \sqrt{I(X)} \cap \sqrt{I(Y)} = I(X) \cap I(Y)$

In general $I(X \cap Y) \neq \underline{I(X) + I(Y)}$

$\qquad\qquad\quad \overset{\shortparallel}{\sqrt{I(X) + I(Y)}}$

$I(X \cap Y) = I(X) + I(Y) \iff I(X) + I(Y)$ is radical

K algebraically closed

$\mathfrak{a} \subsetneq K[x_1 \dots x_n] \Rightarrow V(\mathfrak{a}) \neq \emptyset$

$$(*) \begin{cases} F_1(x_1 \dots x_n) = 0 \\ \vdots \\ F_r(x_1 \dots x_n) = 0 \end{cases}$$

$\exists$ a solution in K^n, K alg. closed,

$\Longleftrightarrow \langle F_1, \dots, F_r \rangle \not\ni 1$

It is clear that if $\underline{1 = H_1 F_1 + \dots + H_r F_r} \Rightarrow$

$(*)$ cannot have solutions.

How to check if $\boxed{1 \in \langle F_1, \dots F_r \rangle}$?

Membership problem: $\mathfrak{a} = \langle F_1, \dots, F_r \rangle$, G

Is $G \in \mathfrak{a}$ or not?

Answer is given by Gröbner bases.

Given $F_1, \dots, F_r$, if $1 \in \langle F_1 \dots F_r \rangle$, can we

bound the degrees of $H_1, \dots, H_r$ s.t. $1 = H_1 F_1 + \dots H_r F_r$?

Effective Nullstellensatz

$X \subseteq \mathbb{P}^n_K$ $\quad I_h(X) = \langle \text{homog. } F \mid F(P) = 0 \ \forall P \in X \rangle$

$V_P(I_h(X)) = \overline{X}$

$\alpha \subseteq K[x_0 \ldots x_n]$ $\quad \alpha$ homog.

$I_h(V_P(\alpha)) \supseteq \sqrt{\alpha}$

radical homog. ideal

Difference between affine and projective ambient:

$\exists$ proper homog. ideals in $K[x_0 \ldots x_n]$ without zeros in $\mathbb{P}^n_K$

$\mathfrak{m} = \langle x_0, \ldots, x_n \rangle$ maximal homog.

$V_P(\mathfrak{m}) = \emptyset$, $\quad V(\mathfrak{m}) \subseteq \mathbb{A}^{n+1}$
$\quad V(\mathfrak{m}) = \{0\}$

$I_h(V_P(\mathfrak{m})) = I_h(\emptyset) = K[x_0 \ldots x_n]$
$I_h(V_P(\mathfrak{m})) \neq \mathfrak{m}$

For proper homog. ideals α st. $V_P(\alpha) = \emptyset$

$I_h(V_P(\alpha)) = K[x_0 \ldots x_n]$

$\sqrt{\alpha} \subsetneq K[x_0 \ldots x_n]$: $1 \in \alpha \Leftrightarrow 1 \in \sqrt{\alpha}$.

What are the proper homog. ideals α in $K[x_0 \dots x_n]$, s.t. $V_P(\alpha) = \emptyset$?

Prop: Weak form of Nullstellensatz in proj. space.
K algebraically closed, $\alpha \subseteq K[x_0 \dots x_n]$ homogeneous
Then:

$$V_P(\alpha) = \emptyset$$

$$\Updownarrow$$

$$\alpha = K[x_0 \dots x_n] \text{ or } \sqrt{\alpha} = \langle x_0, \dots, x_n \rangle = \mathfrak{m} \quad (\Rightarrow V_P(\alpha) = \emptyset)$$

$$\Updownarrow$$

$$\exists\, d \in \mathbb{N} \text{ s.t. } \alpha \supseteq K[x_0 \dots x_n]_d$$

$F \neq 0$, choose a finite set of generators of α

$\alpha = \langle G_1, \ldots, G_s \rangle$ $\qquad V(\alpha) = V(G_1, \ldots, G_s)$

Assumption means: if $P \in \mathbb{A}^n_K$ s.t. $G_1(P) = \cdots = G_s(P) = 0$

then $F(P) = 0$.

"Rabinowitch trick": introduce another variable

$K[x_1 \ldots x_n, t] \supseteq \beta = \langle G_1, \ldots, G_s, \underline{tF - 1} \rangle$

$V(\beta) \subseteq \mathbb{A}^{n+1}$: claim: $V(\beta) = \emptyset$

If $(a_1, \ldots a_n, a_{n+1}) \in V(\beta)$: $\left\{ \begin{array}{l} G_1(a_1 \ldots a_n) = 0 \\ \vdots \\ G_s(a_1 \ldots a_n) = 0 \\ a_{n+1} \underbrace{F(a_1 \ldots a_n)}_{=0} = 1 \end{array} \right.$

$P(a_1, \ldots, a_n)$

<u>contradiction</u>

$V(\beta) = \emptyset \implies \beta = K[x_1 \ldots x_n, t]$

$1 \in \beta$