Hilbert Nullstellensatz $\quad K$ algebraically closed

$\mathfrak{a} \subseteq K[x_1 \ldots x_n]$ $\quad \boxed{I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}}$

$\mathfrak{m} \subseteq K[x_1 \ldots x_n] \quad \mathfrak{m} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$

$\mathfrak{a} \subsetneq K[x_1 \ldots x_n] \Rightarrow V(\mathfrak{a}) \neq \emptyset$

$\mathbb{P}^n_K \quad K[x_0, \ldots, x_n]$, $\mathfrak{a} \subseteq K[x_0, \ldots, x_n]$ homogeneous

If $V_p(\mathfrak{a}) = \emptyset$, $\overset{\mathfrak{a} \neq K[x_0 \ldots x_n]}{\Longrightarrow} I_a(V_p(\mathfrak{a})) \neq \sqrt{\mathfrak{a}}$

$I_a(V_p(\mathfrak{a})) = I_a(\emptyset) = K[x_0 \ldots x_n]$

$\mathfrak{a} \neq K[x_0 \ldots x_n] \Rightarrow \sqrt{\mathfrak{a}} \neq K[x_0 \ldots x_n]$

characterization of homog. ideals of $K[x_0 \ldots x_n]$ s.t. $V_p(\mathfrak{a}) = \emptyset$.

Prop. α homog. ideal in $K[x_0 \ldots x_n]$

$V_p(\alpha) = \emptyset \iff$ either $\alpha = K[x_0 \ldots x_n]$ or $\sqrt{\alpha} = \langle x_0, \ldots, x_n \rangle = M$

$\iff \exists\, d \in \mathbb{N}$ s.t. $\alpha \supseteq K[x_0 \ldots x_n]_d$.

Proof 1) $V_p(\alpha) = \emptyset$ $\quad p: \mathbb{A}^{n+1}_K \setminus \{0\} \longrightarrow \mathbb{P}^n_K$

$V(\alpha) \subseteq \mathbb{A}^{n+1}$

For any homog. id. $\quad V_p(\alpha) = p(V(\alpha) \setminus \{0\})$

If $V_p(\alpha) = \emptyset \implies V(\alpha) \setminus \{0\} = \emptyset$ $\quad V(\alpha) = \emptyset \implies \alpha = K[x_0 \ldots x_n]$; $V(\alpha) = \{0\}$

$V(\alpha) = \{(0, \ldots, 0)\} \implies \underset{\sqrt{\alpha}}{\underset{\shortparallel}{I(V(\alpha))}} = \langle x_0, \ldots, x_n \rangle$

2) Ass. that $\alpha = K[x_0 \ldots x_n] \implies \alpha \supseteq K[x_0 \ldots x_n]_d \quad \forall d$

$\sqrt{\alpha} = \langle x_0, \ldots, x_n \rangle$: $\exists\, d$? s.t. $\alpha \supseteq K[x_0 \ldots x_n]_d$

$\Downarrow$

$\forall i \quad \exists\, r_i \geq 1 \quad x_i^{r_i} \in \alpha$; $d \quad x_0^{i_0} \cdots x_n^{i_n} \quad i_0 + \ldots + i_n = d$

If $\quad i_0 < r_0, \ldots, i_n < r_n \implies \underset{d}{\underset{\shortparallel}{i_0 + \ldots + i_n}} \leq r_0 + \ldots + r_n - (n+1)$

$i_0 \leq r_0 - 1, \ldots, i_n \leq r_n - 1$

Take $\boxed{d \geq r_0 + \ldots + r_n - n} \implies$ in any mon. of deg d, at least one variable x_i occurs w. exponent $\geq r_i \implies \alpha \supseteq K[x_0 \ldots x_n]_d$

3) $\mathfrak{a} \supseteq K[x_0 \dots x_n]_d$, $d \geq 1 \implies$

$\mathfrak{a} \ni x_0^d, \dots, x_n^d \implies V_p(\mathfrak{a}) = \emptyset$ $\square$

Thm. K alg. closed, $\mathfrak{a} \subseteq K[x_0 \dots x_n]$ homog.

F non-constant, homog. pol. s.t. $F \in I_h(V_p(\mathfrak{a}))$

$\implies F \in \sqrt{\mathfrak{a}}$.

Pf. $F \in I_h(V_p(\mathfrak{a})) \implies V_p(F) \supset V_p(\mathfrak{a}) \implies V(F) \setminus \{0\} \supseteq V(\mathfrak{a}) \setminus \{0\}$

$p : \mathbb{A}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$

$$\boxed{V(F) \supseteq V(\mathfrak{a})}$$

$\ni 0$

$$F \in I(V(F)) \subseteq I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$$

Homogenous Nullstellensatz: K alg. closed

$\mathfrak{a} \subseteq K[x_0 \dots x_n]$ homog. s.t. $V_p(\mathfrak{a}) \neq \emptyset$

$\implies I_h(V_p(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Pf. Enough to prove that $\underbrace{I_h(V_p(\mathfrak{a}))}_{\text{homog. ideal}} \subseteq \sqrt{\mathfrak{a}}$

Take F homog. in $I_h(V_p(\mathfrak{a}))$; $V_p(\mathfrak{a}) \neq \emptyset$

$\implies \deg F > 0 \implies F \in \sqrt{\mathfrak{a}}$.

In the proj. case, the analogous of Nullstellensatz involves only homog. ideals having at least one projective zero.

K alg. closed

$$\left\{ \begin{array}{c} \text{Zariski closed subsets} \\ \text{in } \mathbb{P}^n_K, \neq \emptyset \end{array} \right\} \underset{I_h}{\overset{V_P}{\rightleftarrows}} \left\{ \begin{array}{c} \text{homog. radical ideals } \alpha \\ \text{s.t. } V_P(\alpha) \neq \emptyset \end{array} \right\}$$

$$X = V_P(I_h(X))$$

$$\alpha = \sqrt{\alpha} = I_h(V_P(\alpha))$$

Which are the homog. radical ideals w. $V_P(\alpha) = \emptyset$?

$\underline{K[x_0 \ldots x_n]}$ or $\langle x_0, \ldots, x_n \rangle$

maximal irrelevant ideal

$\alpha \subsetneq K[x_0 \ldots x_n]$ w. $\sqrt{\alpha} = \langle x_0 \ldots x_n \rangle$

irrelevant ideal

Integral elements over a ring A

$A \subseteq B$ rings, A subring of B

B is an A-algebra $\Rightarrow$ B can be interpreted as an A-module

Two notions of finiteness for B over A

Def. 1) B is a finite A-algebra is B is finitely generated as A-module

2) B is a finitely generated A-algebra if B is fin. gen. as a ring containing B

1) $\exists\, b_1, \dots, b_n$ s.t. any elem. in B is a linear combination $\boxed{b = a_1 b_1 + \dots + a_n b_n}$, $a_1, \dots, a_n \in A$

$$B = \underbrace{Ab_1}_{\text{A-submodule of B gen. by } b_1} + \dots + Ab_n$$

2) $\exists\, b_1, \dots, b_n \in B$ s.t. B is the minimal ring containing $A, b_1, \dots, b_n$ $\Longleftrightarrow$

$B = A[b_1, \dots, b_n]$: $\forall b \in B$,

$b = F(b_1 \dots b_n)$, $F \in A[x_1 \dots x_n]$

B is isomorphic to $A[x_1 \dots x_n] / I$

$I = \operatorname{Ker} \varphi$ $\qquad \varphi : A[x_1 \dots x_n] \longrightarrow B$

$F(x_1 \dots x_n) \longrightarrow F(b_1 \dots b_n)$

Thm. $A \subseteq B$, $b \in B$, $A[b]$

$A \subseteq A[b] \subseteq B$

1) b is integral over A $\iff$

2) $A[b]$ is a finite A-algebra $\iff$

3) $\exists$ a subring $C \subseteq B$, $A[b] \subseteq C \subseteq B$

s.t. C is a finite A-algebra.

$K \subseteq K'$ $\quad x \in K'$ $\quad x$ is algebraic over K $\iff$

$K[x]$ is a vector space of finite dim d

d = degree of the minimal pol. of x