Chapter 2

Examples of algebraic varieties.

2.1 Points

In the Zariski topology both in $\mathbb{A}^n$ and in $\mathbb{P}^n$ all points are closed. If $P(a_1, \ldots, a_n) \in \mathbb{A}^n$, then $P = V(x_1 - a_1, \ldots, x_n - a_n)$. But in the projective space, if $P[a_0, \ldots, a_n] \in \mathbb{P}^n$, the equations are different: $P = V_P(a_i x_j - a_j x_i)_{i,j=0,\ldots,n}$. In this way the polynomials defining P as closed set are homogeneous. They can be seen as minors of order 2 of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ x_0 & x_1 & \ldots & x_n \end{pmatrix}$$

with entries in $K[x_0, x_1, \ldots, x_n]$. This expresses the fact that $x_0, \ldots, x_n$ are proportional to $a_0, \ldots, a_n$. Equations of the form $V_P(x_0 - a_0, \ldots, x_n - a_n)$ don't make sense.

2.2 Affine and projective linear subspaces.

Generalizing the previous example, the linear subspaces, both in the affine and in the projective case, are examples of algebraic sets. As it is well known, they are defined by equations of degree 1.

2.3 Hypersurfaces

An affine hypersurface is an affine variety of the form $V(F)$, the set of zeros of a unique polynomial F of **positive** degree. Similarly, in the projective space, a projective hypersurface is of the form $V_P(G)$, where $G \in K[x_0, x_1, \ldots, x_n]$ is a homogeneous non-constant polynomial.

Examples of hypersurfaces are the curves in the affine or projective plane, and the surfaces in a space of dimension 3, as for instance the quadrics.

Let us recall that the polynomial ring $K[x_1, \dots, x_n]$ is a UFD (unique factorization domain), i.e., every non-constant polynomial F can be expressed in a unique way (up to the order and up to units) as $F = F_1^{r_1} F_2^{r_2} \dots F_s^{r_s}$, where $F_1, \dots, F_s$ are irreducible and two by two distinct polynomials, and $r_i \geq 1$ for any $i = 1, \dots, s$. Hence the hypersurface of $\mathbb{A}^n$ defined by F is

$$X := V(F) = V(F_1^{r_1} F_2^{r_2} \dots F_s^{r_s}) = V(F_1 F_2 \dots F_s) = V(F_1) \cup V(F_2) \cup \dots \cup V(F_s).$$

The equation $F_1 F_2 \dots F_s = 0$ is called the reduced equation of X. Note that $F_1 F_2 \dots F_s$ generates the radical $\sqrt{F}$. If $s = 1$, X is called an *irreducible* hypersurface; by definition its degree is the degree of its reduced equation. Therefore, any hypersurface is a finite union of irreducible hypersurfaces.

Assume now that $Z = V_P(G)$, with $G \in K[x_0, x_1, \dots, x_n]$, G homogeneous, is a projective hypersurface. Exercise 3 asks to prove that the irreducible factors of G are homogeneous. Therefore, as in the affine case, each projective hypersurface Z has a reduced equation (whose degree is, by definition, the degree of Z) and Z is a finite union of irreducible hypersurfaces.

If the field K is algebraically closed, the degree of a projective hypersurface has the following important geometrical meaning.

Proposition 2.3.1. *Let K be an algebraically closed field. Let $Z \subset \mathbb{P}^n$ be a projective hypersurface of degree d. Then any line in $\mathbb{P}^n$, not contained in Z, meets Z at exactly d points, counting multiplicities.*

In the proof we will see what we mean by saying "counting multiplicity".

Proof. Let G be the reduced equation of Z and $L \subset \mathbb{P}^n$ be any line.

We fix two points on L: $A = [a_0, \dots, a_n], B = [b_0, \dots, b_n]$. So L admits parametric equations of the form

$$\begin{cases} x_0 = \lambda a_0 + \mu b_0 \\ x_1 = \lambda a_1 + \mu b_1 \\ \dots \\ x_n = \lambda a_n + \mu b_n \end{cases}$$

The points of $Z \cap L$ are obtained from the homogeneous pairs $[\lambda, \mu]$ which are solutions of the equation $G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n) = 0$. If $L \subset Z$, then this equation is an identity.

Otherwise, $G(\lambda a_0 + \mu b_0, \ldots, \lambda a_n + \mu b_n)$ is a non-zero homogeneous polynomial of degree d in the two variables λ, μ. Since K is algebraically closed, it can be factorized in linear factors:

$$G(\lambda a_0 + \mu b_0, \ldots, \lambda a_n + \mu b_n) = (\mu_1\lambda - \lambda_1\mu)^{d_1}(\mu_2\lambda - \lambda_2\mu)^{d_2} \ldots (\mu_r\lambda - \lambda_r\mu)^{d_r}$$

with $d_1 + d_2 + \ldots + d_r = d$. Every factor corresponds to a point in $Z \cap L$, to be counted with the same multiplicity as the corresponding factor.

□

If K is not algebraically closed, considering the algebraic closure of K and using Proposition 2.3.1, we get that d is an upper bound on the number of points of $Z \cap L$.

2.4 Product of affine spaces

Let $\mathbb{A}^n$, $\mathbb{A}^m$ be two affine spaces over the field K. The cartesian product $\mathbb{A}^n \times \mathbb{A}^m$ is the set of pairs (P, Q), $P \in \mathbb{A}^n$, $Q \in \mathbb{A}^m$: it is in natural bijection with $\mathbb{A}^{n+m}$ via the map

$$\varphi : \mathbb{A}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^{n+m}$$

such that $\varphi((a_1, \ldots, a_n), (b_1, \ldots, b_m)) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$.

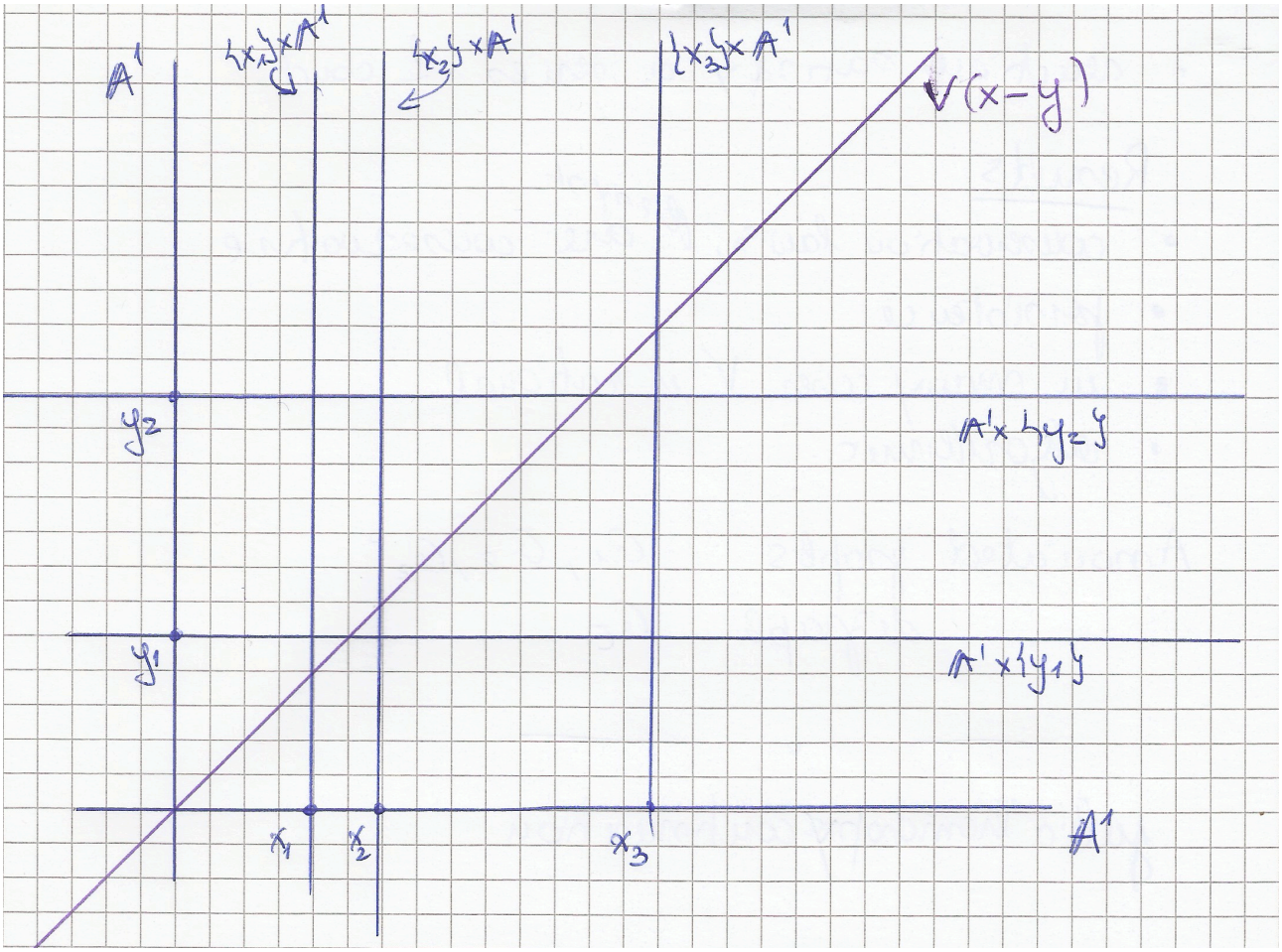
From now on we will always identify $\mathbb{A}^n \times \mathbb{A}^m$ with $\mathbb{A}^{n+m}$. Therefore we have two topologies on $\mathbb{A}^n \times \mathbb{A}^m$: the Zariski topology of $\mathbb{A}^{n+m}$ and the product topology of the Zariski topologies of $\mathbb{A}^n$ and $\mathbb{A}^m$.

Proposition 2.4.1. *The Zariski topology is strictly finer than the product topology.*

Proof. Let us first observe that, if $X = V(\alpha) \subset \mathbb{A}^n$, $\alpha \subset K[x_1, \ldots, x_n]$ and $Y = V(\beta) \subset \mathbb{A}^m$, $\beta \subset K[y_1, \ldots, y_m]$, then $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m$ is Zariski closed, precisely $X \times Y = V(\alpha \cup \beta)$ where the union is made in the polynomial ring in $n + m$ variables $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Now we consider $U = \mathbb{A}^n \setminus X$ and $V = \mathbb{A}^m \setminus Y$, open subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ in the Zariski topology. Then $U \times V = \mathbb{A}^n \times \mathbb{A}^m \setminus ((\mathbb{A}^n \times Y) \cup (X \times \mathbb{A}^m))$: this is a set-theoretical fact that holds true in general. So it is open in $\mathbb{A}^n \times \mathbb{A}^m$ in the Zariski topology.

Conversely, we give an example to prove that the two topologies are different. Precisely we show that $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ contains some subsets which are Zariski open, but are not open in the product topology.

The proper open subsets in the product topology are of the form $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{$ finite unions of "vertical" and "horizontal" lines$\}$. See the figure.


$\mathbb{A}^1$
$\{x_1\} \times \mathbb{A}^1$
$\{x_2\} \times \mathbb{A}^1$
$\{x_3\} \times \mathbb{A}^1$
$V(x-y)$
y_2
$\mathbb{A}^1 \times \{y_2\}$
y_1
$\mathbb{A}^1 \times \{y_1\}$
x_1
x_2
x_3
$\mathbb{A}^1$

Let $X = \mathbb{A}^2 \setminus V(x-y)$: it is Zariski open but does not contain any non-empty subset of the above form, so it is not open in the product topology. There are similar examples in $\mathbb{A}^n \times \mathbb{A}^m$ for any n, m. □

Note that there is no similar construction for $\mathbb{P}^n \times \mathbb{P}^m$. We will see in Chapter 13.1 that there is an injective map, the Segre map, of $\mathbb{P}^n \times \mathbb{P}^m$ to the projective space of dimension $(n+1)(m+1)-1$, whose image is a projective variety. This allows to give a geometric structure to the product of projective spaces . We see here only the first case.

2.5 $\mathbb{P}^1 \times \mathbb{P}^1$

The cartesian product $\mathbb{P}^1 \times \mathbb{P}^1$ is simply a set, but we are going to define an injective map σ from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$, whose image will be a projective variety: it will be identified with our product, and this will allow to interpret $\mathbb{P}^1 \times \mathbb{P}^1$ as a projective variety.

The map σ is defined in the following way: $\sigma([x_0, x_1], [y_0, y_1]) = [x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. Using coordinates $z_0, ..., z_3$ in $\mathbb{P}^3$, σ is defined parametrically by

$$\begin{cases} z_0 = x_0y_0 \\ z_1 = x_0y_1 \\ z_2 = x_1y_0 \\ z_3 = x_1y_1 \end{cases}$$

It is easy to observe that σ is a well–defined map: the image is never $[0,0,0,0]$, and depends uniquely on the pair of points and not on the choice of their coordinates. Moreover σ is injective. Assume that $\sigma([x_0, x_1], [y_0, y_1]) = \sigma([x'_0, x'_1], [y'_0, y'_1])$. Then there exists a non-zero constant λ such that

$$\begin{cases} x_0y_0 = \lambda x'_0y'_0 \\ x_0y_1 = \lambda x'_0y'_1 \\ x_1y_0 = \lambda x'_1y'_0 \\ x_1y_1 = \lambda x'_1y'_1 \end{cases}$$

Now, if $y_0 \neq 0$, then $x_0 = (\lambda y'_0/y_0)x'_0$ and $x_1 = (\lambda y'_0/y_0)x'_1$; if $y_1 \neq 0$, then $x_0 = (\lambda y'_1/y_1)x'_0$ and $x_1 = (\lambda y'_1/y_1)x'_1$; in both cases $[x_0, x_1] = [x'_0, x'_1]$. Similarly one proves that $[y_0, y_1] = [y'_0, y'_1]$.

Let Σ denote the image $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$. It is the quadric of equation $z_0z_3 - z_1z_2 = 0$; indeed, on one hand it is clear that $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset V_P(z_0z_3 - z_1z_2)$; conversely, assume that

$z_0z_3 = z_1z_2$ and $z_0 \neq 0$. Then, multiplying all coordinates by z_0, we get: $[z_0, z_1, z_2, z_3] = [z_0^2, z_0z_1, z_0z_2, z_0z_3]$; by assumption this coincides with $[z_0^2, z_0z_1, z_0z_2, z_1z_2]$, and is therefore equal to $\sigma([z_0, z_2], [z_0, z_1])$. If $z_0 = 0$, the argument is similar, using another non-zero coordinate.

The map σ is called the Segre map and Σ the **Segre variety**. The name comes from the Italian mathematician Corrado Segre (Torino, 1863–1924), the "father" of the Italian school of algebraic geometry.

2.6 Embedding of $\mathbb{A}^n$ in $\mathbb{P}^n$.

We will see now how to unify the two notions introduced so far of affine and projective variety. Precisely, after identifying $\mathbb{A}^n$ with the open subset $U_0 \subset \mathbb{P}^n$ (or with any U_i) (as in Section 1.3), we will prove that the Zariski topology on $\mathbb{A}^n$ coincides with the topology induced by the Zariski topology on $\mathbb{P}^n$.

Let H_i be the hyperplane of $\mathbb{P}^n$ of equation $x_i = 0$, $i = 0, \ldots, n$; it is closed in the Zariski topology, and its complementar set U_i is open. So we have an open covering of $\mathbb{P}^n$: $\mathbb{P}^n = U_0 \cup U_1 \cup \cdots \cup U_n$. Let us recall that for any i there is a bijection $\varphi_i : U_i \to \mathbb{A}^n$ such that $\varphi_i([x_0, \ldots, x_i, \ldots, x_n]) = (\frac{x_0}{x_i}, \ldots, \hat{1}, \ldots, \frac{x_n}{x_i})$. The inverse map is $j_i : \mathbb{A}^n \to U_i$ such that $j_i(y_1, \ldots, y_n) = [y_1, \ldots, 1, \ldots, y_n]$.

Proposition 2.6.1. *The map φ_i is a homeomorphism, for $i = 0, \ldots, n$.*

Proof. Assume $i = 0$ (the other cases are similar).

We introduce two maps:

(i) *dehomogeneization* of polynomials with respect to x_0.

It is a map ${}^a : K[x_0, x_1, \ldots, x_n] \to K[y_1, \ldots, y_n]$ such that

$$ {}^a(F(x_0, \ldots, x_n)) = {}^aF(y_1, \ldots, y_n) := F(1, y_1, \ldots, y_n). $$

Note that ^{a} is a ring homomorphism.

(ii) *homogeneization* of polynomials with respect to x_0.

It is a map ${}^h : K[y_1, \ldots, y_n] \to K[x_0, x_1, \ldots, x_n]$ defined by

$$ {}^h(G(y_1, \ldots, y_n)) = {}^hG(x_0, \ldots, x_n) := x_0^{\deg G} G(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}). $$

${}^hG$ is always a homogeneous polynomial of the same degree as G. The map ^{h} is clearly not a ring homomorphism. Note that always ${}^a({}^hG) = G$ but in general ${}^h({}^aF) \neq F$; what we can say is that, if $F(x_0, \ldots, x_n)$ is homogeneous, then there exists $r \geq 0$ such that $F = x_0^r({}^h({}^aF))$.

Let $X \subset U_0$ be closed in the topology induced by the Zariski topology of the projective space, i.e. $X = U_0 \cap V_P(I)$ where I is a homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$. Define ${}^aI = \{{}^aF \mid F \in I\}$: it is an ideal of $K[y_1, \ldots, y_n]$ (because ^{a} is a ring homomorphism). We prove that $\varphi_0(X) = V({}^aI)$. Indeed, let $P[x_0, \ldots, x_n]$ be a point of U_0; then $\varphi_0(P) = (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in \varphi_0(X) \iff P[x_0, \ldots, x_n] = [1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] \in X = V_P(I) \iff F(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = 0 \ \forall \ {}^aF \in {}^aI \iff \varphi_0(P) \in V({}^aI)$.

Conversely: let $Y = V(\alpha)$ be a Zariski closed subset of $\mathbb{A}^n$, where α is an ideal of $K[y_1, \ldots, y_n]$. Let ${}^h\alpha$ be the homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$ generated by the set $\{{}^hG \mid G \in \alpha\}$. We prove that $\varphi_0^{-1}(Y) = V_P({}^h\alpha) \cap U_0$. Indeed $[1, x_0, \ldots, x_n] \in \varphi_0^{-1}(Y) \iff (x_1, \ldots, x_n) \in Y \iff G(x_1, \ldots, x_n) = {}^hG(1, x_1, \ldots, x_n) = 0 \ \forall \ G \in \alpha \iff [1, x_1, \ldots, x_n] \in V_P({}^h\alpha)$. □

From now on we will often identify $\mathbb{A}^n$ with U_0 via φ_0 (and similarly with U_i via φ_i). So if $P[x_0, \ldots, x_n] \in U_0$, we will refer to $x_0, \ldots, x_n$ as the homogeneous coordinates of P and to $\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}$ as the non–homogeneous or affine coordinates of P.

Exercises 2.6.2. It will be useful to remember that any algebraically closed field is infinite.

1. Assume that K is an algebraically closed field.

   a) Prove that, if $n \geq 1$, then in $\mathbb{A}^n_K$ the complementar set of any hypersurface has infinitely many points.

   b) Prove that, if $n \geq 2$, then also any hypersurface has infinitely many points.

2. Prove that the Zariski topology on $\mathbb{A}^n$ is T_1.

3. Let $F \in K[x_0, x_1, \ldots, x_n]$ be a homogeneous polynomial. Check that its irreducible factors are homogeneous. (Hint: prove that a product of two polynomials not both homogeneous is not homogeneous.)

Solution of Exercise 1.

Let the hypersurface in question be defined by $F(x_1, \ldots, x_n) = 0$, F non constant. We can assume that the variable x_n occurs in F. So we have an expression

$$F = f_0 + f_1 x_n + \cdots + f_d x_n^d,$$

with $f_i \in K[x_1, \ldots, x_{n-1}]$ $\forall i$, $d > 0$ and $f_d \neq 0$.

a) For this first part it is enough to assume that K is an infinite field. We proceed by induction on the number of variables. If $n = 1$, the statement is true because K is infinite. Let $n > 1$: by the inductive assumption, there exist infinitely many $(a_1, \ldots, a_{n-1}) \in K^{n-1}$ such that $f_d(a_1, \ldots, a_{n-1}) \neq 0$. Then for any such $(n-1)$-tuple $F(a_1, \ldots, a_{n-1}, x_n)$ is a non-zero polynomial of degree $d > 0$ in $K[x_n]$: it has finitely many zeros, so there are infinitely many $a_n \in K$ such that $F(a_1, \ldots, a_{n-1}, a_n) \neq 0$.

b) As in a), there exist infinitely many $(a_1, \ldots, a_{n-1}) \in K^{n-1}$ such that $f_d(a_1, \ldots, a_{n-1}) \neq 0$. Since K is algebraically closed, for each of these $(a_1, \ldots, a_{n-1})$ there is at least one $a_n \in K$ such that $F(a_1, \ldots, a_{n-1}, a_n) = 0$.

Chapter 3

The ideal of an algebraic set and the Hilbert Nullstellensatz.

3.1 The ideal of an algebraic set

Let $X \subset \mathbb{A}^n$ be an affine variety, $X = V(\alpha)$, where $\alpha \subset K[x_1, \ldots, x_n]$ is an ideal.

The ideal α defining X is not unique. We have already made this observation in the case of the hypersurfaces (Section 2.3). For another example, let $O = \{(0,0)\} \subset \mathbb{A}^2$ be the origin; then $O = V(x_1, x_2) = V(x_1^2, x_2) = V(x_1^2, x_2^3) = V(x_1^2, x_1x_2, x_2^2) = \ldots$ Nevertheless, there is an ideal we can canonically associate to X: the biggest one among the ideals defining it.

We give the following definition:

Definition 3.1.1. Let $Y \subset \mathbb{A}^n$ be any set. The *ideal of* Y is

$$I(Y) = \{F \in K[x_1, \ldots, x_n] \mid F(P) = 0 \text{ for any } P \in Y\} = \{F \in K[x_1, \ldots, x_n] \mid Y \subset V(F)\} :$$

it is the set of **all** polynomials vanishing on Y. Note that $I(Y)$ is in fact an ideal, because the sum of two polynomials vanishing along Y also vanishes along Y, and the product of any polynomial by a polynomial vanishing along Y again vanishes along Y.

Example 3.1.2. Maximal ideal of a point. If $P(a_1, \ldots, a_n)$ is a point, then $I(P) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$. Indeed all the polynomials of $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ vanish on P, and moreover it is a maximal ideal.

The fact that $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ is maximal can be understood looking at the quotient ring $K[x_1, \ldots, x_n]/\langle x_1 - a_1, \ldots, x_n - a_n \rangle$: the idea is that in the quotient the variables $x_1, \ldots, x_n$ are replaced by the constants $a_1, \ldots, a_n$, so it has to be $K[a_1, \ldots, a_n] = K$. Since the quotient is a field, the ideal is maximal.

Another proof of the maximality of $\langle x_1 - a_1, \ldots, x_n - a_n \rangle$ can be given by exploiting the expansion in power series around $\underline{a} := (a_1, \ldots, a_n)$ of any polynomial $F(x_1, \ldots, x_n)$. I first recall that this expansion is possible for polynomials over any field, without involving any differentiation process, but using only the formal definition of derivative for polynomials. See for instance [W], pp. 21-23.

The proof goes as follows. Assume that $F(a_1, \ldots, a_n) = 0$ and use the Taylor expansion:

$$F(x_1, \ldots, x_n) = F(\underline{a}) + \sum_{i=1}^{n} (x_i - a_i) F_{x_i}(\underline{a}) + \sum_{i,j=1}^{n} (x_i - a_i)(x_j - a_j) F_{x_i x_j}(\underline{a}) + \ldots$$

It follows that $F \in \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

Remark 1. The following relations follow immediately by the definition:

(i) if $Y \subset Y'$, then $I(Y) \supset I(Y')$;

(ii) $I(Y \cup Y') = I(Y) \cap I(Y')$;

(iii) $I(Y \cap Y') \supset I(Y) + I(Y')$.

In the projective ambient, we have an analogous situation.

Definition 3.1.3. If $Z \subset \mathbb{P}^n$ is any set, the *homogeneous ideal of* Z is, by definition, the homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$ generated by the set

$$\{G \in K[x_0, x_1, \ldots, x_n] \mid G \text{ is homogeneous and } V_P(G) \supset Z\}.$$

It is denoted $I_h(Z)$.

Relations similar to (i),(ii),(iii) of Remark 1 are satisfied. $I_h(Z)$ is also the set of polynomials $F(x_0, \ldots, x_n)$ such that every point of Z is a projective zero of F.

If $X = V(\alpha)$ we want to understand the relation between α and $I(X)$. Let $\alpha \subset K[x_1, \ldots, x_n]$ be an ideal. Let $\sqrt{\alpha}$ denote the radical of α:

$$\sqrt{\alpha} =: \{F \in K[x_1, \ldots, x_n] \mid \exists r \geq 1 \ s.t. \ F^r \in \alpha\}.$$

Note that $\sqrt{\alpha}$ is an ideal (why?) and that always $\alpha \subset \sqrt{\alpha}$; if equality holds, then α is called a *radical ideal*.

Proposition 3.1.4. *The ideal of a subset of the affine space is radical. More precisely:*

1. *for any $X \subset \mathbb{A}^n$, $I(X)$ is a radical ideal;*

2. for any $Z \subset \mathbb{P}^n$, $I_h(Z)$ is a homogeneous radical ideal.

Proof. 1. If $F \in \sqrt{I(X)}$, let $r \geq 1$ such that $F^r \in I(X)$: if $P \in X$, then $(F^r)(P) = 0 = (F(P))^r$ in the base field K. Therefore $F(P) = 0$.

2. is similar, taking into account that $I_h(Z)$ is a homogeneous ideal (see Exercise 6).

□

We can interpret I as a map from $\mathcal{P}(\mathbb{A}^n)$, the power set of the affine space, to $\mathcal{P}(K[x_1, \ldots, x_n])$, the power set of the polynomial ring. On the other hand, V can be seen as a map in the opposite sense. We have:

Proposition 3.1.5. *Let $\alpha \subset K[x_1, \ldots, x_n]$ be an ideal, let $Y \subset \mathbb{A}^n$ be any subset. Then:*

(i) $\alpha \subset I(V(\alpha))$;

(ii) $Y \subset V(I(Y))$;

(iii) $V(I(Y)) = \overline{Y}$: the closure of Y in the Zariski topology of $\mathbb{A}^n$.

Proof. (i) If $F \in \alpha$ and $P \in V(\alpha)$, then $F(P) = 0$, so $F \in I(V(\alpha))$.

(ii) If $P \in Y$ and $F \in I(Y)$, then $F(P) = 0$, so $P \in V(I(Y))$.

(iii) Taking closures in (ii), we get: $\overline{Y} \subset \overline{V(I(Y))} = V(I(Y))$, because it is already closed. Conversely, let $X = V(\beta)$ be any closed set containing Y: $X = V(\beta) \supset Y$. Then $I(Y) \supset I(V(\beta)) \supset \beta$ by (i); we apply V again: $V(\beta) = X \supset V(I(Y))$ so any closed set containing Y contains $V(I(Y))$ so $\overline{Y} \supset V(I(Y))$.

□

Similar properties relate homogeneous ideals of $K[x_0, x_1, \ldots, x_n]$ and subsets of $\mathbb{P}^n$; in particular, if $Z \subset \mathbb{P}^n$, then $V_P(I_h(Z)) = \overline{Z}$, the closure of Z in the Zariski topology of $\mathbb{P}^n$. In the projective case, one has to take care of the fact that any homogeneous ideal is generated by the set of its homogeneous elements, and so, to prove an inclusion between homogeneous ideals, it is enough to check it on the homogeneous elements.

3.2 Nullstellensatz

There is no characterization of $I(V(\alpha))$ **in general**. We can only say that it is a radical ideal containing α, so it contains also $\sqrt{\alpha}$. To characterise $I(V(\alpha))$ we have to put the properties of the base field K into play.

The following celebrated theorem gives the answer for algebraically closed fields.

Theorem 3.2.1 (Hilbert's Nullstellensatz - Theorem of zeros). *Let K be an algebraically closed field. Let $\alpha \subset K[x_1, \ldots, x_n]$ be an ideal. Then $I(V(\alpha)) = \sqrt{\alpha}$.*

Remark 2. The assumption on K is necessary. Let me recall that K is algebraically closed if any non–constant polynomial of $K[x]$ has at least one root in K, or, equivalently, if any irreducible polynomial of $K[x]$ has degree 1. So if K is not algebraically closed, there exists an irreducible polynomial $F \in K[x]$ of degree $d > 1$. Therefore F has no zeros in K, hence $V(F) \subset \mathbb{A}^1_K$ is empty. So $I(V(F)) = I(\emptyset) = \{G \in K[x] \mid \emptyset \subset V(G)\} = K[x]$. But $\langle F \rangle$ is a maximal ideal in $K[x]$, and $\langle F \rangle \subset \sqrt{\langle F \rangle}$. If $\langle F \rangle \neq \sqrt{\langle F \rangle}$, by the maximality $\sqrt{\langle F \rangle} = \langle 1 \rangle$, so $\exists r \geq 1$ such that $1^r = 1 \in \langle F \rangle$, which is false. Hence $\sqrt{\langle F \rangle} = \langle F \rangle \neq K[x] = I(V(F))$.

We will deduce the proof of Hilbert Nullstellensatz, after several steps, from another very important theorem, known as "Emmy Noether normalization Lemma".

We start with some definitions.

Let $K \subset E$ be fields, K subfield of E. Let $\{z_i\}_{i \in I}$ be a family of elements of E.

Definition 3.2.2. The family $\{z_i\}_{i \in I}$ is *algebraically free* over K or, equivalently, the elements z_i's are *algebraically independent* over K if there is no non–zero polynomial $F \in K[x_i]_{i \in I}$, the polynomial ring in a set of variables indexed on I, that vanishes in the elements of the family $\{z_i\}$.

For example: if the family consists of only one element z, $\{z\}$ is algebraically free over K if and only if z is transcendental over K. The family $\{\pi, \sqrt{\pi}\}$ is not algebraically free over $\mathbb{Q}$: it satisfies the non–trivial relation $x_1^2 - x_2 = 0$.

By convention, the empty family is free over any field K.

Let $\mathcal{S}$ be the set of the families of elements of E, that are algebraically free over K. $\mathcal{S}$ is a non–empty set, partially ordered by inclusion and inductive. By Zorn's lemma, $\mathcal{S}$ has maximal elements, i.e. algebraically free families that do not remain free if any element of E is added. Any such maximal algebraically free family is called a *transcendence basis* of E over K. It can be proved that, if B, B' are two transcendence bases, then they have the same cardinality, called the *transcendence degree* of E over K. It is denoted $tr.d.E/K$.

Definition 3.2.3. A *K–algebra* is a ring A containing (a subfield isomorphic to) K.

Let $y_1, \dots, y_n$ be elements of E: the *K–algebra generated by $y_1, \dots, y_n$* is, by definition, the minimum subring of E containing $K, y_1, \dots, y_n$: it is denoted $K[y_1, \dots, y_n]$ and its elements are polynomials in the elements $y_1, \dots, y_n$ with coefficients in K. Its quotient field $K(y_1, \dots, y_n)$ is the minimum subfield of E containing $K, y_1, \dots, y_n$.

A *finitely generated K–algebra A* is a K–algebra containing finitely many elements $y_1, \dots, y_n$ such that $A = K[y_1, \dots, y_n]$.

Given elements $y_1, \dots, y_n$ in an extension E of K, we can consider the **evaluation homomorphism** from the polynomial ring in n variables to the K-algebra generated by $y_1, \dots, y_n$

$$\varphi : K[x_1, \dots, x_n] \to K[y_1, \dots, y_n] \text{ such that } F(x_1, \dots, x_n) \to F(y_1, \dots, y_n). \tag{3.1}$$

The kernel of φ is formed by the polynomials vanishing at the n-tuple $(y_1, \dots, y_n)$. Therefore φ is injective if and only if $y_1, \dots, y_n$ are algebraically independent over K, if and only if φ gives an isomorphism between the K-algebra $K[y_1, \dots, y_n]$ and the polynomial ring in n variables.

Remark 3. A K-algebra A is finitely generated if and only if A is isomorphic to a quotient of a polynomial ring in finitely many variables over K. Indeed, if $A = K[y_1, \dots, y_n]$, considering the evaluation map φ (3.1), from the homomorphism theorem it follows that $A \simeq K[x_1, \dots, x_n]/\ker\varphi$. Conversely, given a quotient $A = K[x_1, \dots, x_n]/\alpha$, let $\xi = [x_i]$ be the equivalence class of the variable x_i in A. Then any element of A can be written as a polynomial $F(\xi_1, \dots, \xi_n)$, therefore A is the K-algebra generated by $\xi_1, \dots, \xi_n$.

Proposition 3.2.4. *$K(y_1, \dots, y_n)$ has a transcendence basis over K contained in the set $\{y_1, \dots, y_n\}$.*

Proof. Let $\mathcal{S}$ be the set of all subfamilies of $\{y_1, \dots, y_n\}$ formed by algebraically independent elements: $\mathcal{S}$ is a finite set so it has maximal elements with respect to the inclusion. We can assume that $\{y_1, \dots, y_r\}$ is such a maximal family. Then $y_{r+1}, \dots, y_n$ are all algebraic over $K(y_1, \dots, y_r)$ so $K(y_1, \dots, y_n)$ is an algebraic extension of $K(y_1, \dots, y_r)$. If $z \in K(y_1, \dots, y_n)$ is any element, then z is algebraic over $K(y_1, \dots, y_r)$, so the family $\{y_1, \dots, y_r, z\}$ is not algebraically free. □

Corollary 3.2.5. *$tr.d.K(y_1, \dots, y_n)/K \leq n$.*

Let now $A \subset B$ be rings, A a subring of B.

Definition 3.2.6. Let $b \in B$: b is *integral over* A if it is a root of a monic polynomial of $A[x]$, i.e., there exist $a_1, \ldots, a_n \in A$ such that

$$b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_n = 0.$$

Such a relation is called an integral equation, or an equation of integral dependence, for b over A.

Note that, if A is a field, then b is integral over A if and only if b is algebraic over A.

Definition 3.2.7. B is called *integral over* A, or, B is an integral extension of A, if any $b \in B$ is integral over A.

We can state now the

Theorem 3.2.8. Normalization Lemma. *Let A be a finitely generated K–algebra and an integral domain. Let $r := tr.d.K(y_1, \ldots, y_n)/K$. Then there exist elements $z_1, \ldots, z_r \in A$, algebraically independent over K, such that A is integral over $K[z_1, \ldots, z_r]$.*

Proof. We postpone the proof to Chapter 4 . □

We start now the proof of the Nullstellensatz.

1^{st} **Step.**

Let K be an algebraically closed field, let $\mathcal{M} \subset K[x_1, \ldots, x_n]$ be a maximal ideal. Then, there exist $a_1, \ldots, a_n \in K$ such that $\mathcal{M} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

Proof. Let K' be the quotient ring $K[x_1, \ldots, x_n]/\mathcal{M}$: it is a field because $\mathcal{M}$ is maximal, and it is a K–algebra finitely generated by the residues in K' of $x_1, \ldots, x_n$. By the Normalization Lemma, there exist $z_1, \ldots, z_r \in K'$, algebraically independent over K, such that K' is integral over $A := K[z_1, \ldots, z_r]$. We claim that A *is a field*: let $f \in A$, $f \neq 0$; $f \in K'$ so there exists $f^{-1} \in K'$, and f^{-1} is integral over A; we fix an integral equation for f^{-1} over A:

$$(f^{-1})^s + a_{s-1}(f^{-1})^{s-1} + \cdots + a_0 = 0$$

where $a_0, \ldots, a_{s-1} \in A$. We multiply this equation by f^{s-1}:

$$f^{-1} + a_{s-1} + \cdots + a_0 f^{s-1} = 0$$

hence $f^{-1} \in A$. So A is both a field and a polynomial ring over K, so $r = 0$ and $A = K$. Therefore K' is an algebraic extension of K, which is algebraically closed, so $K' \simeq K$. Let

us fix an isomorphism $\psi : K' = K[x_1, \dots, x_n]/\mathcal{M} \xrightarrow{\sim} K$ and let $p : K[x_1, \dots, x_n] \to K' = K[x_1, \dots, x_n]/\mathcal{M}$ be the canonical epimorphism.

Let $a_i = \psi(p(x_i))$, $i = 1, \dots, n$. The kernel of $\psi \circ p$ is $\mathcal{M}$, and $x_i - a_i \in \ker(\psi \circ p)$ for any i. So $\langle x_1 - a_1, \dots, x_n - a_n \rangle \subset \ker(\psi \circ p) = \mathcal{M}$. Since $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is maximal (see Example 3.1.2), we conclude the proof of the 1^{st} Step.

2^{nd} **Step** (Weak Nullstellensatz).

Let K be an algebraically closed field, let $\alpha \subsetneq K[x_1, \dots, x_n]$ be a *proper* ideal. Then $V(\alpha) \neq \emptyset$ i.e. the polynomials of α have at least one common zero in $\mathbb{A}^n_K$.

Proof. Since α is proper, there exists a maximal ideal $\mathcal{M}$ containing α. Then $V(\alpha) \supset V(\mathcal{M})$. By 1^{st} Step, $\mathcal{M} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$, so $V(\mathcal{M}) = \{P\}$ with $P(a_1, \dots, a_n)$, hence $P \in V(\alpha)$. For any maximal ideal containing α we get a point in $V(\alpha)$.

3^{rd} **Step** (Rabinowitch method or Rabinowitch trick).

Let K be an algebraically closed field: we will prove that $I(V(\alpha)) \subset \sqrt{\alpha}$. Since the reverse inclusion always holds, this will conclude the proof.

Let $F \in I(V(\alpha))$, $F \neq 0$ (if $F = 0$ the conclusion is clear, because each ideal contains 0), and let $\alpha = \langle G_1, \dots, G_r \rangle$. The assumption on F means: if P is a point such that $G_1(P) = \cdots = G_r(P) = 0$, then $F(P) = 0$. The Rabinowitch trick consists in introducing an extra variable, and then specializing it. Let us consider the polynomial ring in $n+1$ variables $K[x_1, \dots, x_{n+1}]$ and let β be the ideal $\beta = \langle G_1, \dots, G_r, x_{n+1}F - 1 \rangle$: clearly by assumption β has no zeros in $\mathbb{A}^{n+1}$, hence, by Step 2, $1 \in \beta$, i.e. there exist $H_1, \dots, H_{r+1} \in K[x_1, \dots, x_{n+1}]$ such that

$$1 = H_1G_1 + \cdots + H_rG_r + H_{r+1}(x_{n+1}F - 1).$$

This is an equality of polynomials, so equality still holds if we give to some of the variables a special value. In particular we can specialize the new variable x_{n+1} replacing it with $\frac{1}{F}$. More formally, we introduce the K-homomorphism $\psi : K[x_1, \dots, x_{n+1}] \to K(x_1, \dots, x_n)$ defined by $H(x_1, \dots, x_{n+1}) \to H(x_1, \dots, x_n, \frac{1}{F})$.

The polynomials $G_1, \dots, G_r$ do not contain x_{n+1} so $\psi(G_i) = G_i \ \forall \ i = 1, \dots, r$. Moreover $\psi(x_{n+1}F - 1) = 0$, $\psi(1) = 1$. Therefore

$$1 = \psi(H_1G_1 + \cdots + H_rG_r + H_{r+1}(x_{n+1}F - 1)) = \psi(H_1)G_1 + \cdots + \psi(H_r)G_r$$

where $\psi(H_i)$ is a rational function with denominator a power of F. By multiplying this relation by a common denominator, that is a power of F, we get an expression of the form:

$$F^m = H'_1G_1 + \cdots + H'_rG_r,$$

so $F \in \sqrt{\alpha}$. □

Corollary 3.2.9. *Let K be an algebraically closed field.*

1. *There is a bijection between the algebraic subsets of $\mathbb{A}^n$ and the radical ideals of $K[x_1, \ldots, x_n]$. The bijection is given by $\alpha \to V(\alpha)$ and $X \to I(X)$. In fact, if X is closed in the Zariski topology, then $V(I(X)) = X$; if α is a radical ideal, then $I(V(\alpha)) = \alpha$.*

2. *Let $X, Y \subset \mathbb{A}^n$ be Zariski closed sets. Then*

   (i) $I(X \cap Y) = \sqrt{I(X) + I(Y)}$;

   (ii) $I(X \cup Y) = I(X) \cap I(Y) = \sqrt{I(X)I(Y)}$.

3. *The points of a hypersurface determine its reduced equation.*

Proof. 1. is clear. 2. follows from next Lemma 3.2.10, using the Nullstellensatz. To prove 3., assume that F, G are square-free polynomials in $K[x_1 \ldots, x_n]$ such that $V(F) = V(G)$. Notice that if F is square-free, the $\langle F \rangle = \sqrt{F}$. By the Nullstellensatz it follows that $\sqrt{F} = I(V(F)) = I(V(G)) = \sqrt{G}$, so $\langle F \rangle = \langle G \rangle$, which means that F, G differ at most by units. □

Lemma 3.2.10. *Let α, β be ideals of $K[x_1, \ldots, x_n]$. Then*

a) $\sqrt{\sqrt{\alpha}} = \sqrt{\alpha}$;

b) $\sqrt{\alpha + \beta} = \sqrt{\sqrt{\alpha} + \sqrt{\beta}}$;

c) $\sqrt{\alpha \cap \beta} = \sqrt{\alpha\beta} = \sqrt{\alpha} \cap \sqrt{\beta}$.

Proof. a) if $F \in \sqrt{\sqrt{\alpha}}$, there exists $r \geq 1$ such that $F^r \in \sqrt{\alpha}$, hence there exists $s \geq 1$ such that $F^{rs} \in \alpha$.

b) $\alpha \subset \sqrt{\alpha}$, $\beta \subset \sqrt{\beta}$ imply $\alpha + \beta \subset \sqrt{\alpha} + \sqrt{\beta}$ hence $\sqrt{\alpha + \beta} \subset \sqrt{\sqrt{\alpha} + \sqrt{\beta}}$.

Conversely, $\alpha \subset \alpha + \beta$, $\beta \subset \alpha + \beta$ imply $\sqrt{\alpha} \subset \sqrt{\alpha + \beta}$, $\sqrt{\beta} \subset \sqrt{\alpha + \beta}$, hence $\sqrt{\alpha} + \sqrt{\beta} \subset \sqrt{\alpha + \beta}$ so $\sqrt{\sqrt{\alpha} + \sqrt{\beta}} \subset \sqrt{\sqrt{\alpha + \beta}} = \sqrt{\alpha + \beta}$.

c) $\alpha\beta \subset \alpha \cap \beta \subset \alpha$ (resp. $\subset \beta$) therefore $\sqrt{\alpha\beta} \subset \sqrt{\alpha \cap \beta} \subset \sqrt{\alpha} \cap \sqrt{\beta}$. If $F \in \sqrt{\alpha} \cap \sqrt{\beta}$, then $F^r \in \alpha$, $F^s \in \beta$ for suitable $r, s \geq 1$, hence $F^{r+s} \in \alpha\beta$, so $F \in \sqrt{\alpha\beta}$.

□

Part 2.*(i)* of Corollary 3.2.9 implies that $I(X \cap Y) = I(X) + I(Y)$ if and only if $I(X) + I(Y)$ is a radical ideal (see Remark 1 (iii)).

Remark 4. The weak form of the Nullstellensatz says that a system of algebraic equations has at least one solution over an algebraically closed field if, and only if, the ideal generated by the corresponding polynomials is proper, or, equivalently, if it is impossible to find a linear combination of them, with coefficients in the polynomial ring, equal to the constant 1. The proof of Nullstellensatz we have given is not constructive, in the sense that, given polynomials $F_1, \ldots, F_r$, it does not say how to check if 1 belongs or not to the ideal $\langle F_1, \ldots, F_r \rangle$.

The problem of making the proof constructive is connected to the more general "ideal membership problem", which asks, given an ideal $\alpha \subset K[x_1 \ldots, x_n]$ and a polynomial $G \in K[x_1 \ldots, x_n]$, to decide if $G \in \alpha$ or not.

Answers to these problems can be given with the tools of computational algebra, in particular using the theory of Gröbner bases. There are effective versions of the Nullstellensatz that allow to bound the degrees of the coefficients in a possible expression $1 = H_1F_1 + \cdots + H_rF_r$, depending on the degrees of $F_1, \ldots, F_r$, and hence to reduce the question to a problem in linear algebra.

3.3 Homogeneous Nullstellensatz

We move now to the projective space. There exist *proper* homogeneous ideals of $K[x_0, x_1, \ldots, x_n]$ without zeros in $\mathbb{P}^n$, even assuming K algebraically closed: for example the maximal ideal $\langle x_0, x_1, \ldots, x_n \rangle$. For such an ideal I, the Nullstellensatz fails, indeed $I_h(V_P(I)) = I_h(\emptyset) = K[x_0, \ldots, x_n]$, but $\sqrt{I} \neq K]x_0, \ldots, x_n]$, because $1 \in I$ if and only if $1 \in \sqrt{I}$.

The following characterization holds:

Proposition 3.3.1. *Let K be an algebraically closed field and let I be a homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$.*

The following are equivalent:
(i) $V_P(I) = \emptyset$;
(ii) either $I = K[x_0, x_1, \ldots, x_n]$ or $\sqrt{I} = \langle x_0, x_1, \ldots, x_n \rangle$;
(iii) there exists $d \geq 1$ such that $I \supset K[x_0, x_1, \ldots, x_n]_d$, the homogeneous component of $K[x_0, x_1, \ldots, x_n]$ of degree d.

Proof. (i)⇒(ii) Let $p : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the canonical surjection. We have: $V_P(I) = p(V(I) - \{0\})$, where $V(I) \subset \mathbb{A}^{n+1}$. So if $V_P(I) = \emptyset$, then either $V(I) = \emptyset$ or $V(I) = \{0\}$. If $V(I) = \emptyset$ then $I(V(I)) = I(\emptyset) = K[x_0, x_1, \ldots, x_n]$; if $V(I) = \{0\}$, then $I(V(I)) = \langle x_0, x_1, \ldots, x_n \rangle = \sqrt{I}$ by the Nullstellensatz.

(ii)⇒(iii) Let $\sqrt{I} = K[x_0, x_1, \ldots, x_n]$, then $1 \in \sqrt{I}$ so $1^r = 1 \in I (r \geq 1)$. If $\sqrt{I} = \langle x_0, x_1, \ldots, x_n \rangle$, then for any variable x_k there exists an index $i_k \geq 1$ such that $x_k^{i_k} \in I$. If $d \geq i_0 + i_1 + \cdots + i_n - n$, then any monomial of degree d is in I, so $K[x_0, x_1, \ldots, x_n]_d \subset I$.

(iii)⇒(i) because no point in $\mathbb{P}^n$ has all coordinates equal to 0. □

Theorem 3.3.2. *Let K be an algebraically closed field and I be a homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$. If F is a homogeneous non–constant polynomial such that $V_P(F) \supset V_P(I)$ (i.e. F vanishes on $V_P(I)$, or $F \in I_h(V_P(I))$), then $F \in \sqrt{I}$.*

Proof. We have $p(V(I) - \{0\}) = V_P(I) \subset V_P(F)$. Since F is non–constant, we have also $V(F) = p^{-1}(V_P(F)) \cup \{0\}$, so $V(F) \supset V(I)$; by the Nullstellensatz $I(V(I)) = \sqrt{I} \supset I(V(F)) = \sqrt{(F)} \ni F$. □

Corollary 3.3.3 (homogeneous Nullstellensatz)**.** *Let I be a homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$ such that $V_P(I) \neq \emptyset$, K algebraically closed. Then $\sqrt{I} = I_h(V_P(I))$.*

Definition 3.3.4. A homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$ such that $\sqrt{I} = \langle x_0, x_1, \ldots, x_n \rangle$ is called *irrelevant*.

Corollary 3.3.5. *Let K be an algebraically closed field. There is a bijection between the set of projective algebraic subsets of $\mathbb{P}^n$ and the set of radical homogeneous non–irrelevant ideals of $K[x_0, x_1, \ldots, x_n]$.*

Remark. Let $X \subset \mathbb{P}^n$ be an algebraic set, $X \neq \emptyset$. The affine cone of X, denoted by $C(X)$, is the following subset of $\mathbb{A}^{n+1}$: $C(X) = p^{-1}(X) \cup \{0\}$, where $p : (K^{n+1})^* \to \mathbb{P}^n$ is the canonical projection (see Section 1.2). If $X = V_P(F_1, \ldots, F_r)$, with $F_1, \ldots, F_r$ homogeneous, then $C(X) = V(F_1, \ldots, F_r)$. By the Nullstellensatz, if K is algebraically closed, $I(C(X)) = I_h(X)$.

Exercises 3.3.6.

1. Give a non-trivial example of an ideal α of $K[x_1, \ldots, x_n]$ such that $\alpha \neq \sqrt{\alpha}$.

2. Let K be an algebraically closed with char $K \neq 2$. Show that the following closed subsets of the affine plane are such that equality does not hold in the relation $I(Y \cap Y') \supset I(Y) + I(Y')$: $Y = V(x^2 + y^2 - 1)$ and $Y' = V(y - 1)$.

3. Let $\alpha \subset K[x_1, \ldots, x_n]$ be an ideal. Prove that $\alpha = \sqrt{\alpha}$ if and only if the quotient ring $K[x_1, \ldots, x_n]/\alpha$ does not contain any non–zero nilpotent.

4. Consider $\mathbb{Z} \subset \mathbb{Q}$. Prove that if an element $y \in \mathbb{Q}$ is integral over $\mathbb{Z}$, then $y \in \mathbb{Z}$. (Hint: fixed $y = a/b \in \mathbb{Q}$ integral over $\mathbb{Z}$, write an integral equation for y, then use the unique factorization in $\mathbb{Z}$.)

5. Let us recall that a prime ideal of a ring R is an ideal $\mathcal{P}$ such that $a \notin \mathcal{P}$, $b \notin \mathcal{P}$ implies $ab \notin \mathcal{P}$. Prove that any prime ideal is a radical ideal.

6. * Let I be a homogeneous ideal of $K[x_1, \ldots, x_n]$ satisfying the following condition: *if F is a homogeneous polynomial such that $F^r \in I$ for some positive integer r, then $F \in I$.* Prove that I is a radical ideal. (Hint: take F non homogeneous such that for some $r \geq 1$ $F^r \in I$, then use induction on the number of non-zero homogeneous components of F to prove that $F \in I$.)