Fondamenti di Automatica

19/03/2024

Esercizi su Laplace-trasf e Z trasf.

$$Y(z) = \frac{2}{z(z-1)} \longrightarrow y(k) = ?$$

1° modo

$$\frac{Y(z)}{z} = \frac{2}{z^2(z-1)}$$

$$= \frac{C_{1,1}}{z} + \frac{C_{1,2}}{z^2} + \frac{C_2}{z-1}$$

$$C_2 = \lim_{z \to 1} (z-1)\left[\frac{Y(z)}{z}\right] = \lim_{z \to 1} \cancel{(z-1)} \frac{2}{z^2\cancel{(z-1)}} = 2$$

$$C_{1,2} = \lim_{z \to 0} z^2\left[\frac{Y(z)}{z}\right] = \lim_{z \to 0} \cancel{z^2} \frac{2}{\cancel{z^2}(z-1)} = -2$$

$$C_{1,1} = \lim_{z \to 0} \frac{d}{dz}\left\{z^2\left[\frac{Y(z)}{z}\right]\right\} =$$

$$\frac{Y(z)}{z} = \frac{2}{z^2(z-1)} = \frac{C_{1,1}}{z} - \frac{2}{z^2} + \frac{2}{z-1}$$

$$= \frac{C z(z-1) - 2(z-1) + 2z^2}{z^2(z-1)}$$

$$2 = Cz^2 + 2z^2 - Cz - 2z + 2$$

$$= (C+2)z^2 - (C+2)z + 2$$

$$C+2 = 0$$
$$C+2 = 0 \longrightarrow C_{1,1} = -2$$

$$\frac{Y(z)}{z} = -\frac{2}{z} - \frac{2}{z^2} + \frac{2}{z-1} \quad \Big/ \cdot z$$

$$Y(z) = -2 - 2\cdot(z^{-1}) + 2\cdot\frac{z}{z-1}$$

$$-2\cdot 1 \quad -2\cdot 1\cdot z^{-1}$$

ztref

$$1 \longleftrightarrow \delta(k)$$

$$-2\cdot\delta(k) \qquad -2\delta(k-1) + 2\cdot 1(k)$$

$$y(k) = -2\delta(k) - 2\delta(k-1) + 2\cdot 1(k)$$

$$Y(z) = \frac{2}{z(z-1)} \qquad N(z) \quad m=0 \qquad n-m=2$$
$$D(z) \quad n=2$$

$$y(0)=0 \quad y(1)=0 \quad y(2)=+2$$

$$y(k) \xrightarrow{k\to+\infty} ?$$

$$Y(z) = \frac{2}{z(z-1)} \xrightarrow[\text{tvf}]{} \lim_{k\to+\infty} y(k) = \lim_{z\to 1} \frac{z-1}{z} Y(z)$$

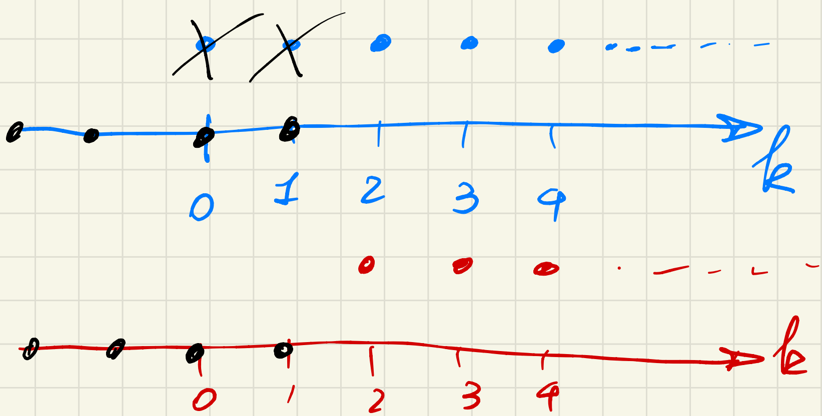
val finale
OK

$$\lim_{z\to 1} \cancel{\frac{z-1}{z}} \cdot \frac{2}{z\cancel{(z-1)}} = +2$$

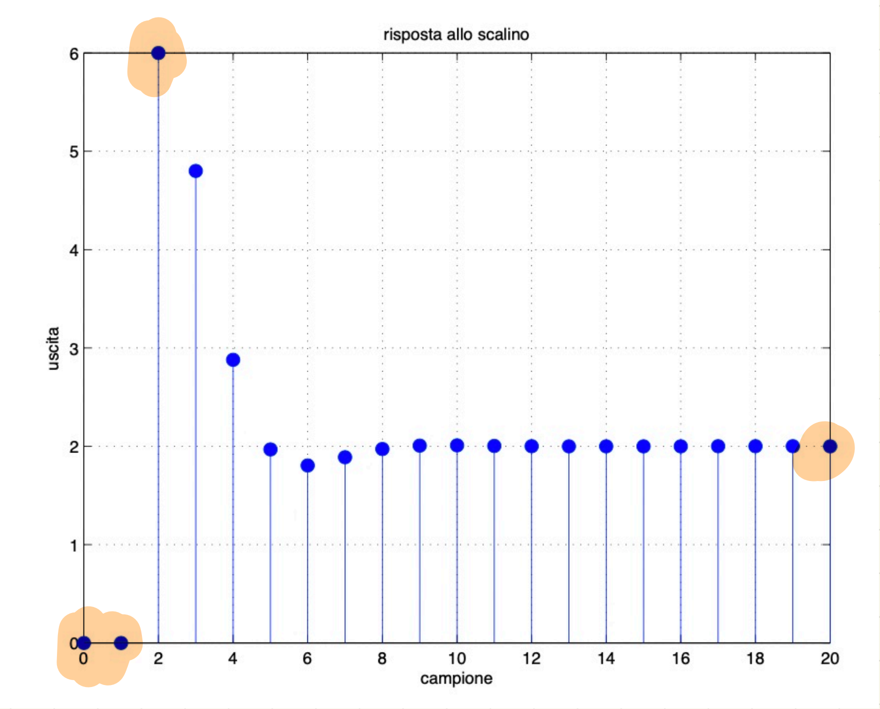
$$Y(z) = \frac{2}{z(z-1)} = 2 \cdot \frac{z}{z-1} \cdot (z^{-2})$$

$$y(k) = 2 \cdot 1(k-2)$$

$$y(k) = 2 \cdot 1(k) - 2\delta(k) - 2\delta(k-1)$$



30/6/2014



$y(k)$

$$\boxed{\begin{gathered} y(0) = y(1) = 0 \qquad y(2) = +6 \\ y(k) \xrightarrow[k \to \infty]{} +2 \end{gathered}}$$

$$Y_A(z) = \frac{30z - 24}{(5z^2 - 3z + 1)(z-1)} \qquad Y_B(z) = \frac{30}{(5z^2 - 3z + 1)(z-1)}$$

$$Y_C(z) = \frac{2z^2(5z-4)}{5\left(z - \frac{1}{10}\right)\left(z + \frac{1}{2}\right)(z-1)}$$

$$Y_B(z) = \frac{30}{(5z^2-3z+1)(z-1)}$$

$m=0$

$n=3$

NO

$y(0)=y(1)=y(2)$

$y(3)=\frac{30}{5}=+6$

$$Y_C(z) = \frac{2z^2(5z-4)}{5\left(z-\frac{1}{10}\right)\left(z+\frac{1}{2}\right)(z-1)}$$

$m=3$

$n=3$

NO

$$y(0)=\frac{2\cdot 5}{5}=+2$$

$$\boxed{Y_A(z)} = \frac{6(5z-4)}{(5z^2-3z+1)(z-1)}$$

$m=1$

$n=3$

$n-m=2$

$y(0)=y(1)=0$

$$y(2)=\frac{30}{5}=+6$$

$$\lim_{z\to 1}\frac{z-1}{z}\cdot Y_A(z)=\lim_{z\to 1}\frac{6(5z-4)}{z(5z^2-3z+1)}=\frac{6\cdot 1}{1(5-3+1)}=+2$$

30/6/2014

Dato il sistema LTI a tempo discreto, descritto dalle eq. di stato

$$
\begin{cases}
x_1(k+1) = 2x_1(k) + 2x_2(k) + u(k) \\
x_2(k+1) = 3x_1(k) + 7x_2(k) + 3u(k) \\
y(k) = -x_1(k)
\end{cases}
$$

l'evoluzione libera dell'uscita y(k)

Determinare l'espressione analitica di $y(k)$ quando al sistema si applica l'ingresso

$$u(k) = 0 \cdot 1(k)$$

e le condizioni iniziali sono

a partire dalle condizioni $X(0) = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$

Voglio utilizzare la Z trasformata

$$\mathcal{Z}\{x_1(k+1)\} = 2X_1(z) + 2X_2(z) + \cancel{U(z)}^{\,0}$$

$$z\left[X_1(z) - \underset{+1}{x_1(0)}\right] = 2X_1(z) + 2X_2(z)$$

$$\mathcal{Z}\{x_2(k+1)\} = 3X_1(z) + 7X_2(z) + \cancel{U(z)}^{\,0}$$

$$z\left[X_2(z) - \underset{-1}{x_2(0)}\right] = \dots$$

$$\begin{cases} zX_1(z) = z + 2X_1(z) + 2X_2(z) \\ zX_2(z) = -z + 3X_1(z) + 7X_2(z) \end{cases}$$

$$(z-2)X_1(z) = z + 2X_2(z)$$

$$X_1(z) = \frac{z}{z-2} + \frac{2}{z-2}X_2(z)$$

$$X_1(z) = \frac{z}{z-2} + \frac{2}{z-2} X_2(z)$$

$$X_2(z) = \frac{-z}{z-7} + \frac{3}{z-7} X_1(z)$$

$$= -\frac{z}{z-7} + \frac{3}{z-7}\left[\frac{z}{z-2} + \frac{2}{z-2} X_2(z)\right]$$

$$\left[1 - \frac{6}{(z-7)(z-2)}\right] X_2(z) = \frac{3z}{(z-7)(z-2)} - \frac{z}{z-7}$$

$$X_2(z) = \frac{3z - z(z-2)}{\cancel{(z-7)}\cancel{(z-2)}} \cdot \frac{\cancel{(z-7)}\cancel{(z-2)}}{(z-1)(z-8)}$$

$$= \frac{z(5-z)}{(z-1)(z-8)} = X_2(z)$$

$$X_1(z) = \frac{z(z-9)}{(z-1)(z-8)}$$

$$Y(z) = -X_1(z) = -\frac{z(z-9)}{(z-1)(z-8)}$$

$m=2$

$n=2$

$$\frac{Y(z)}{z} = -\frac{z-9}{(z-1)(z-8)} = \frac{C_1}{z-1} + \frac{C_2}{z-8}$$

$$C_1 = \lim_{z \to 1} (z-1)\left[\frac{Y(z)}{z}\right] = -8/7$$

$$C_2 = \lim_{z \to 8} (z-8)\left[\frac{Y(z)}{z}\right] = 1/7$$

$$Y(z) = -\frac{8}{7} \cdot \frac{z}{z-1} + \frac{1}{7} \cdot \frac{z}{z-8}$$

$$\frac{z}{z-1} \xrightarrow{\mathcal{Z}^{-1}} 1(k) \qquad \frac{z}{z-8} \xrightarrow{\mathcal{Z}^{-1}} 8^k \cdot 1(k)$$

$$y(k) = \left[-\frac{8}{7} + \frac{1}{7} \cdot 8^k\right] \cdot 1(k)$$

$$y(0) = -\frac{8}{7} + \frac{1}{7} = -1$$

FA 5/02/2021

Determinare i coefficienti della L-trasformata

$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{s\left[a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0\right]}$$

che corrisponde alla funzione causale $y(t)$ con caratteristiche

- presenza di oscillazioni smorzate
- $y(0) = 0$
- $\dot{y}(0) = 0$
- $\ddot{y}(0) = +10$
- $y(t) \xrightarrow[t \to +\infty]{} 0$

$$Y(s) = \frac{N(s)}{s \cdot D(s)}$$

$N(s) \to$ grado m

$D(s) \to$ grado n

(a) comportamento oscillante smorzato $\to$ poli compl. coniugati

$$p_i = \sigma_i + j\omega_i$$

$$\sigma_i < 0$$

$(\bullet)\quad y(0) = 0$

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s)$$

$$= \lim_{s\to\infty} \cancel{s}^{1} \cdot \frac{N(s)}{\cancel{s}_{1} \cdot D(s)} = 0$$

$n > m$

$(\bullet)\quad \dot{y}(0) = 0$

$$\lim_{t\to 0} \dot{y}(t) = \lim_{s\to+\infty} s\left[sY(s) - \cancel{y(0)}_{0}\right]$$

$$= \lim_{s\to\infty} \cancel{s^2}^{s} \cdot \frac{N(s)}{\cancel{s}_{1} \cdot D(s)} = \lim_{s\to+\infty} \frac{\overbrace{sN(s)}^{m+1}}{\underbrace{D(s)}_{n}} = 0$$

$n > m+1$

$(\bullet)\quad \ddot{y}(0) = +10$

$$\lim_{t\to 0} \ddot{y}(t) = \lim_{s\to\infty} \cancel{s}^{1}\left[s^2 \frac{N(s)}{\cancel{s}_{1} D(s)}\right] = +10$$

$$= \lim_{s\to+\infty} \frac{s^2 N(s)}{D(s)} = +10$$

$n = m+2$

$$\frac{b_m}{a_n} = +10$$

$(\cdot)$ $\lim_{t \to +\infty} y(t) = 0$

$$Y(s) = \frac{N(s)}{s D(s)}$$

$D(0) \neq 0$

$p : D(p) = 0$ $\quad \forall p : Re(p) < 0$

$$\lim_{t \to +\infty} y(t) = \lim_{s \to 0} \cancel{s} \cdot \frac{N(s)}{\cancel{s} D(s)} = \frac{N(0)}{D(0)} = 0$$

$$= \frac{b_0}{a_0} = 0$$

$N(0) = 0$

$$Y(s) = \frac{N(s)}{s \cdot D(s)}$$

m

n

$n = m + 2$

$$\frac{b_m}{a_n} = +10$$

$b_0 = 0$

$m = 1 \quad N(s)$

$\downarrow$

$n = +3 \quad D(s)$

$D(s)$ poli c.c.

$Re < 0$

$$Y(s) = \frac{10\, s}{s \cdot \left[(s+1)\left(s^2+s+16\right)\right]}$$

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FA 5/02/21

Si consideri il sistema LTI a tempo continuo descritto dalle eq. di stato

$$\begin{cases} \dot{x}_1(t) = u(t) \\ \dot{x}_2(t) = \frac{1}{2} u(t) \\ \dot{x}_3(t) = -\frac{1}{5} x_3(t) + \frac{1}{5} u(t) \\ y(t) = x_3(t) \end{cases}$$

Determinare il movimento forzato dell'uscita quando

$$u(t) = 1(t-2) - 1(t-5)$$

e ovviamente $x(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\mathcal{L}\{\dot{x}_1(t)\} = U(s)$$

$$sX_1(s) - \cancel{x_1(0)}^{\,0} = U(s)$$

$$\mathcal{L}\{\dot{x}_2(t)\} = \frac{1}{2} U(s)$$

$$\mathcal{L}\{\dot{x}_3(t)\} = -\frac{1}{5} X_3(s) + \frac{1}{5} U(s)$$

$$\begin{cases} sX_1(s) - \cancel{x_1(0)}^{0} = U(s) \\ sX_2(s) - \cancel{x_2(0)}^{0} = \frac{1}{2}U(s) \\ sX_3(s) - \cancel{x_3(0)}^{0} = -\frac{1}{5}X_3(s) + \frac{1}{5}U(s) \end{cases}$$

$$U(s) = \mathcal{L}\{1(t-2) - 1(t-5)\}$$

$$= \mathcal{L}\{1(t-2)\} - \mathcal{L}\{1(t-5)\}$$

$$\mathcal{L}\{1(t)\} = \frac{1}{s} \qquad \mathcal{L}\{1(t-2)\} = \frac{1}{s}e^{-2s}$$

$$\mathcal{L}\{1(t-5)\} = \frac{1}{s}e^{-5s}$$

$$\mathcal{L}\{(t-1)\cdot 1(t-1)\} = ? \qquad \frac{1}{s^2}e^{-s}$$

$$\mathcal{L}\{t\cdot 1(t)\} = \frac{1}{s^2}$$

$$\mathcal{L}\{(t-2)\cdot 1(t-1)\} = ?$$

$$(t-2)\cdot 1(t-1) = (t-1)\cdot 1(t-1) - 1(t-1)$$

$$U(s) = \frac{1}{s}\left[e^{-2s} - e^{-5s}\right]$$

$$\begin{cases} X_1(s) = U(s) \\ X_2(s) = \frac{1}{2} U(s) \\ X_3(s) = \frac{1/5}{s+1/5} \cdot U(s) \\ \boxed{Y(s) = X_3(s) = \frac{1/5}{s+1/5} U(s)} \end{cases}$$

$$Y(s) = \underbrace{\frac{1/5}{(s+1/5)s}}\left[e^{-2s} - e^{-5s}\right]$$

$$Y_1(s) \xleftrightarrow{\mathcal{L}^{-1}} y_1(t)$$

$$y(t) = y_1(t-2) - y_1(t-5)$$

$$Y_1(s) = \frac{1/5}{s(s+\frac{1}{5})} = \frac{C_1}{s} + \frac{C_2}{s+\frac{1}{5}}$$

$$C_1 = \lim_{s \to 0} s\, Y_1(s) = \lim_{s \to 0} \cancel{s} \cdot \frac{1/5}{\cancel{s}\left(s + 1/5\right)} = +1$$

$$C_2 = \lim_{s \to -\frac{1}{5}} \left(s + \frac{1}{5}\right) Y_1(s) = \lim_{s \to -\frac{1}{5}} \cancel{\left(s + \frac{1}{5}\right)} \frac{1/5}{s\cancel{\left(s + 1/5\right)}} = -1$$

$$Y_1(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{5}}$$

$$\frac{1}{s} \xrightarrow{\mathcal{L}^{-1}} 1(t) \qquad \frac{1}{s+\frac{1}{5}} \xrightarrow{\mathcal{L}^{-1}} e^{-t/5} \cdot 1(t)$$

$$y_1(t) = \left[1 - e^{-t/5}\right] \cdot 1(t)$$

$$y(t) = \left[1 - e^{-(t-2)/5}\right] \cdot 1(t-2) - \left[1 - e^{-(t-5)}\right] \cdot 1(t-5)$$

FA 17/9/2020

Si consideri il sistema nonlineare a tempo continuo, descritto dalle eq di stato

$$\begin{cases} \dot{x}_1(t) = x_2(t) + [2 - x_1(t)] \cdot u(t) \\ \dot{x}_2 = x_1(t) + [2 + 4\, x_2(t)] \cdot u(t) \\ y(t) = [1 - x_1(t)] \cdot [1 - x_2(t)] \cdot [1 - u(t)] \end{cases}$$

(a) Determinare l'ingresso di equilibrio $\bar{u}$ tale che in corrispondenza di $u(t) \equiv \bar{u}$ lo stato

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

sia stato di eq. del sistema

(b) Determinare le matrici del sistema linearizzato

(a)

$$\dot{x}(t) = f[x(t), u(t)]$$

$\bar{x}$ stato di eq

$\bar{u}$

$$0 = f(\bar{x}, \bar{u})$$

$$\left.\begin{matrix}\dot{x}_1 = 0 \\ \dot{x}_2 = 0\end{matrix}\right|_{x=\bar{x}} \qquad \begin{cases} 0 = \bar{x}_2 + (2-\bar{x}_1)\bar{u} \\ 0 = \bar{x}_1 + 2(1+2\bar{x}_2)\bar{u} \end{cases}$$

$$\begin{cases} \bar{x}_1 = 0 \\ \bar{x}_2 = 0 \end{cases} \Rightarrow \bar{u} = ?$$

$$\begin{cases} 2\bar{u} = 0 \\ 2\bar{u} = 0 \end{cases} \qquad \bar{u} = 0$$

per $u(t) \equiv \bar{u} = 0$ $\quad \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ è stato di equilibrio

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(b)

$$A = \left[\frac{\partial f_i}{\partial x_j}\right]_{\bar{x},\bar{u}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{\bar{x},\bar{u}}$$

$$\begin{cases} \dot{x}_1 = x_2 + (2-x_1)\cdot u \\ \dot{x}_2 = x_1 + 2(1+2x_2)u \end{cases}$$

$$y = (1-x_1)(1-x_2)(1-u)$$

$$\frac{\partial f_1}{\partial x_1} = -\bar{u} \qquad \frac{\partial f_1}{\partial x_2} = +1$$

$$\frac{\partial f_2}{\partial x_1} = +1 \qquad \frac{\partial f_2}{\partial x_2} = 4\bar{u}$$

$$A = \begin{bmatrix} -\bar{u} & +1 \\ +1 & 4\bar{u} \end{bmatrix}_{\bar{u}=0} = \begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix}_{\bar{x},\bar{u}} = \begin{bmatrix} 2-\bar{x}_1 \\ 2(1+2\bar{x}_2) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$C = \left[\frac{\partial g_i}{\partial x_j}\right]_{\bar{x},\bar{u}} = \begin{bmatrix} -(1-\bar{x}_2)(1-\bar{u}) \\ -(1-\bar{x}_1)(1-\bar{u}) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$D = \left[\frac{\partial g_i}{\partial u_j}\right]_{\bar{x},\bar{u}} = -(1-\bar{x}_1)(1-\bar{x}_2) = [-1]$$

FA 20/07/2020

Si consideri il sistema a tempo discreto

$$\begin{cases} x_1(k+1) = 9x_1(k) - 2x_2(k) + 8u(k) \\ x_2(k+1) = 4x_1(k) \\ y = 1{,}5\,x_1(k) - 3x_2(k) \end{cases}$$

Determinare stato ed uscita di equilibrio in corrispondenza dell'ingresso $u(k) = \bar{u} = 2$

$\bar{x}_1 = x_1(k) = x_1(k+1) \ \forall k$

$\bar{x}_2 = x_2(k) = x_2(k+1) \ \forall$

$$\begin{cases} \bar{x}_1 = 9\bar{x}_1 - 2\bar{x}_2 + 8\bar{u} \\ \bar{x}_2 = 4\bar{x}_1 \end{cases} \qquad \bar{u} = 2$$

$$\begin{cases} 8\bar{x}_1 - 2\bar{x}_2 = 16 \\ \bar{x}_2 = 4\bar{x}_1 \end{cases} \rightarrow \quad 0 = 16 \ !!$$

$\nexists$ stato d'eq.!

$$A = \begin{bmatrix} 9 & -2 \\ 4 & 0 \end{bmatrix}$$

$$\det(I - A) \neq 0$$

$$\det(I - A) = 0$$

$$\{\lambda_A\} = \{+1, \dots\}$$

$$\det(\lambda I - A) = 0 \qquad \det \begin{bmatrix} (\lambda - 9) & +2 \\ -4 & \lambda \end{bmatrix} = 0$$

$$(\lambda - 9)\lambda + 8 = 0 \qquad \lambda^2 - 9\lambda + 8 = 0$$

$$\lambda_{1,2} = \frac{9 \pm 7}{2} \quad \begin{cases} +8 \\ \boxed{+1} \end{cases}$$

λ! stab eq.