5 Lesson 5 – March 15^{th}, 2021

5.1 Signed and complex measures

The content of this paragraph can be find in [2, §19].

Definition 11. *Let $(\Omega, \mathcal{A})$ a measurable space (Ω is a set, $\mathcal{A}$ a σ-algebra of subset of Ω). Let $\nu : \mathcal{A} \to]-\infty, +\infty]$ (or $[-\infty, +\infty[$). ν is a signed measure if*

i) $\nu(\emptyset) = 0$;

ii) ν is countably additive.

Remark 6. *In the above definition, ν is countably additive means the following. Let $(A_n)_n$ be a sequence in $\mathcal{A}$ consisting of pairwise disjoint subsets of Ω. Let $A = \bigcup_n A_n$.*

i) If $\nu(A) < +\infty$, then $\sum_n |\nu(A_n)| < +\infty$ and $\sum_n \nu(A_n) = \nu(A)$.

ii) If $\nu(A) = +\infty$, then, denoting by

$$B_n = \begin{cases} A_n & \text{if } \nu(A_n) > 0, \\ \emptyset & \text{if } \nu(A_n) \leq 0, \end{cases} \qquad C_n = \begin{cases} \emptyset & \text{if } \nu(A_n) > 0, \\ A_n & \text{if } \nu(A_n) \leq 0, \end{cases}$$

we have $\sum_n \nu(B_n) = +\infty$ and $\sum_n -\nu(C_n) < +\infty$.

Definition 12. *Let $(\Omega, \mathcal{A})$ a measurable space. Let $\nu : \mathcal{A} \to \mathbb{C}$. ν is a complex measure if*

i) $\nu(\emptyset) = 0$;

ii) ν is countably additive.

Remark 7. *In the above definition, ν is countably additive means the following. Let $(A_n)_n$ be a sequence in $\mathcal{A}$ consisting of pairwise disjoint subsets of Ω. Let $A = \bigcup_n A_n$. Then $\sum_n |\nu(A_n)| < +\infty$ and $\sum_n \nu(A_n) = \nu(A)$.*

Theorem 15. *Let ν be a signed measure. We have*

i) if E, $F \in \mathcal{A}$, $F \subseteq E$ and $|\nu(E)| < +\infty$, then $|\nu(F)| < +\infty$;

ii) if $(A_n)_n$ is sequence in $\mathcal{A}$ with, for all n, $A_n \subseteq A_{n+1}$, then

$$\nu(\bigcup_n A_n) = \lim_n \nu(A_n);$$

iii) if $(A_n)_n$ is sequence in $\mathcal{A}$ with, for all n, $A_n \supseteq A_{n+1}$ and $|\nu(A_1)| < +\infty$, then

$$\nu(\bigcap_n A_n) = \lim_n \nu(A_n).$$

Remark 8. *A result similar to Theorem 15 is valid also for complex measures.*

Theorem (monotonicity of the measure)

Let $(\Omega, \mathcal{A})$ a measurable space Ω set, $\mathcal{A}$ σ-algebra and let ν be a signed measure. Then

i) if $E, F \in \mathcal{A}$ and $E \subseteq F$ and $|\nu(F)| < +\infty$ then also $|\nu(E)| < +\infty$

ii) if $(A_n)_n$ sequence in $\mathcal{A}$ with $A_n \subseteq A_{n+1}$ ("increasing" sequence) with $A = \bigcup_n A_n$ then $\nu(A) = \lim_n \nu(A_n)$

iii) if (A_n) in $\mathcal{A}$ with $A_n \supseteq A_{n+1}$ ("decreasing" sequence) with $A = \bigcap_n A_n$

if $|\nu(A_1)| < +\infty$ then $\lim_n \nu(A_n) = \nu(A)$.

proof. Exercise (from the notion of countable additivity)

Rem. similar result for complex measures

5.2 The Hahn's decomposition theorem

The content of this paragraph can be find in [2, §19].

Let Ω be a set and $\mathcal{A}$ a σ-algebra on Ω. Let ν be a signed measure on the measurable space $(\Omega, \mathcal{A})$.

Definition 13. *Let $P,\ N \in \mathcal{A}$. The couple (P, N) is called a Hahn's decomposition of the measure ν if*

i) $P \cap N = \emptyset$ and $P \cup N = \Omega$;

ii) for all $A \in \mathcal{A}$, $\nu(A \cap P) \geq 0$, (we will say that P is non negative);

iii) for all $A \in \mathcal{A}$, $\nu(A \cap N) \leq 0$, (we will say that N is non positive).

Lemma 7. *Let $E \in \mathcal{A}$, with $-\infty < \nu(E) < +\infty$.*

Then, for all $\varepsilon > 0$, there exists $E_\varepsilon \in \mathcal{A}$ such that

$$E_\varepsilon \subseteq E,\ \nu(E_\varepsilon) \geq \nu(E) \text{ and, for all } A \in \mathcal{A}, \text{ if } A \subseteq E_\varepsilon, \text{ then } \nu(A) \geq -\varepsilon.$$

Proof. By contradiction, suppose that there exists $\varepsilon_0 > 0$ such that, for all $F \in \mathcal{A}$, the fact that $F \subseteq E$ and $\nu(F) \geq \nu(E)$ implies that there exists $A_0 \in \mathcal{A}$ such that $A_0 \subseteq F$ and $\nu(A_0) < -\varepsilon_0$.

Let's choose firstly $F = E$. We have that there exists $A_1 \in \mathcal{A}$ such that $A_1 \subseteq E$ and $\nu(A_1) < -\varepsilon_0$. Choose $F = E \setminus A_1$. Consequently there exists $A_2 \in \mathcal{A}$ such that $A_2 \subseteq E \setminus A_1$ and $\nu(A_2) < -\varepsilon_0$. Next choose $F = E \setminus (A_1 \cup A_2)$. We obtain $A_3 \in \mathcal{A}$ such that $A_3 \subseteq E \setminus (A_1 \cup A_2)$ and $\nu(A_3) < -\varepsilon_0$.

A sequence of pairwise disjoint sets $(A_n)_n$ is constructed in such a way that, for all n, $A_n \subseteq E$ and $\nu(A_n) < -\varepsilon_0$. We deduce that $\nu(\cup_n A_n) = \sum_n \nu(A_n) = -\infty$, obtaining a contradiction, since $\cup_n A_n \subseteq E$ and $\nu(E) > -\infty$. $\square$

Lemma 8. *Let $E \in \mathcal{A}$, with $-\infty < \nu(E) < +\infty$.*

Then there exists $F \in \mathcal{A}$ such that

$$F \subseteq E,\ \nu(F) \geq \nu(E) \text{ and, for all } A \in \mathcal{A},\ \nu(A \cap F) \geq 0$$

(remark that the last point means that F is a non negative set).

Proof. We apply Lemma 7 to the set E, with $\varepsilon = -1$. We obtain that there exists $E_1 \in \mathcal{A}$, such that $E_1 \subseteq E$, $\nu(E_1) \geq \nu(E)$ and, for all $A \in \mathcal{A}$, if $A \subseteq E_1$ then $\nu(A) \geq -1$. Next we apply Lemma 7 to the set E_1 with $\varepsilon = -\frac{1}{2}$. We obtain that there exists $E_2 \in \mathcal{A}$, such that $E_2 \subseteq E_1$, $\nu(E_2) \geq \nu(E_1)$ and, for all $A \in \mathcal{A}$, if $A \subseteq E_2$ then $\nu(A) \geq -\frac{1}{2}$.

Applying successively this procedure, we construct a sequence $(E_n)_n$ such that, for all n, $E_n \subseteq \cdots \subseteq E_1 \subseteq E$, $\nu(E_n) \geq \cdots \geq \nu(E_1) \geq \nu(E)$ and, for all $A \in \mathcal{A}$, if $A \subseteq E_n$ then $\nu(A) \geq -\frac{1}{n}$.

To conclude the proof it is sufficient to take $F = \cap_n E_n$. It is easy to verify that $F \subseteq E$, $\nu(F) \geq \nu(E)$ and, for all $A \in \mathcal{A}$, if $A \subseteq F$ then $\nu(A) \geq 0$, since, for all n, $A \subseteq E_n$ and consequently, for all n, $\nu(A) \geq -\frac{1}{n}$. $\square$

Theorem 16 (Hahn's decomposition theorem). *Let ν be a signed measure on the measurable space $(\Omega, \mathcal{A})$.*

Then there exists a Hahn's decomposition (P, N) of the measure ν. If (P', N') is another Hahn decomposition, then the sets $P \setminus P'$, $P' \setminus P$, $N \setminus N'$ and $N' \setminus N$ are negligeable.

Hahn's decomposition theorem.

def. Let $(\Omega, \mathcal{F})$ as before. Let ν a signed measure.

Let $P, N \in \mathcal{F}$

the couple (P, N) is said a Hahn's decomposition of ν if

i) $P \cap N = \emptyset$, $P \cup N = \Omega$

ii) $\forall A \in \mathcal{F}$, $\nu(P \cap A) \geq 0$, $\nu(N \cap A) \leq 0$

Ex. $(\Omega, \mathcal{F}, \lambda)$ measure space, λ positive measure,

consider $f \in L^1_\lambda(\Omega)$ $\quad (f : \Omega \to \mathbb{R})$

consider $\nu_f : \mathcal{F} \to \mathbb{R}$, $\quad \nu_f(A) = \int_A f \, d\lambda$

ν_f is a <u>signed measure</u> (verify this statement as an exercise)

$\nu_f(\emptyset) = 0$ and <u>countably</u> additivity

Hahn's decomposition?

$P = \{x \in \Omega : f(x) \geq 0\}$

$N = \{x \in \Omega : f(x) < 0\}$

H decomp.

$A \quad \int_{A \cap P} f$ on $A \cap P$, $f \geq 0$

so that $\nu_f(A \cap P) \geq 0$

Theorem (Hahn's decomposition)

Let $(\Omega, \mathcal{A})$ as before. Let ν signed measure.
Then $\exists\ (P, N)$ Hahn's decomposition.
And this is unique up to sets of measure 0.
(i.e. if $(\tilde{P}, \tilde{N})$ is another Hahn's decomp.
then $\nu(P \setminus \tilde{P}) = \nu(N \setminus \tilde{N}) = \nu(\tilde{P} \setminus P) = \nu(\tilde{N} \setminus N) = 0$)

Lemma +. Let $(\Omega, \mathcal{A}, \nu)$ as before. Let $E \in \mathcal{A}$
with $-\infty < \nu(E) < +\infty$
then $\forall \varepsilon > 0\ \exists E_\varepsilon \in \mathcal{A}$ s.t.
(*) $E_\varepsilon \subseteq E$, $\nu(E_\varepsilon) \geq \nu(E)$, $\forall A \in \mathcal{A}$,
if $A \subseteq E_\varepsilon$ then $\nu(A) \geq -\varepsilon$.

proof.
by contradiction. Suppose (*) is not true.

$\neg$(*)
$\exists \varepsilon_0$ s.t. $\forall F \in \mathcal{A}$ it is

$$\neg\big(\underbrace{F \subseteq E,\ \nu(F) \geq \nu(E)}_{p},\ \underbrace{\forall A \in \mathcal{A}\ \text{if } A \subseteq F \text{ then } \nu(A) \geq -\varepsilon}_{q}\big)$$

$\neg(p \wedge q) \iff \neg p \vee \neg q$ $\qquad (\neg p \vee h \iff p \Rightarrow h)$
$\Updownarrow$
$p \Rightarrow \neg q$

$\neg q \iff \exists A \in \mathcal{A} : A \subseteq F \wedge \nu(A) < -\varepsilon$

$$\neg(*) \iff \underbrace{\exists \varepsilon_0 > 0 : \forall F \in \mathcal{A},\ \text{if } F \subseteq E \text{ and } \nu(F) \geq \nu(E) \text{ then } \exists A \in \mathcal{A} : A \subseteq F \wedge \nu(A) < -\varepsilon_0}_{(**)}$$

We apply (**) with $E = F$

$E \subseteq E$ and $\nu(E) \geq \nu(E)$ then
$\exists A_0$ s.t. $A_0 \subseteq E$ and $\nu(A_0) \leq -\varepsilon_0$

we apply (**) with $F = E \setminus A_0$
we have $E \setminus A_0 \subseteq E$ and $\nu(E \setminus A_0) = \nu(E) - \underbrace{\nu(A_0)}_{\geq \varepsilon_0 > 0} \geq \nu(E)$

so $\exists A_1$, $A_1 \subseteq E \setminus A_0$ and $\nu(A_1) < -\varepsilon_0$

we apply (**) with $F = E \setminus (A_0 \cup A_1)$ and so on
$\nu(A_1) \leq -\varepsilon_0$

so we construct $(A_n)_n$, with $A_n \subseteq E$, $A_n \cap A_k = \emptyset$ if $n \neq k$

with $\nu(A_n) \leq -\varepsilon_0\ \forall n$
this means that $\nu(\bigcup_n A_n) = -\infty$

and $\bigcup_n A_n \subseteq E$
but $|\nu(E)| < +\infty$ impossible

Lemma 2 $(\Omega, \mathcal{F}, \nu)$ as before. Let $E \in \mathcal{F}$ with $-\infty < \nu(E) < +\infty$

Then $\exists F \in \mathcal{F}$ s.t. $F \subseteq E$, $\nu(F) \geq \nu(E)$ and $\forall A \in \mathcal{F}, \quad \nu(F \cap A) \geq 0$

F is a non negative set.

proof.

We have $-\infty < \nu(E) < +\infty$

Apply Lemma 1 to E with $\varepsilon_0 = -1$

We obtain $\exists E_1 \in \mathcal{F}$ s.t. $E_1 \subseteq E$, $\nu(E_1) \geq \nu(E)$ and $\forall A \in \mathcal{F}$ if $A \subseteq E_1$ then $\nu(A) \geq -1$

remark that $E_1 \subseteq E$ so $-\infty < \nu(E_1) < +\infty$

we apply Lemma 1 to E_1 with $\varepsilon = -\frac{1}{2}$

$\exists \; E_2 \subseteq E_1 (\subseteq E)$ with $\nu(E_2) \geq \nu(E_1) (\geq \nu(E))$ s.t. $\forall A \in \mathcal{F}$, $A \subseteq E_2$ then $\nu(A) \geq -\frac{1}{2}$

we apply Lemma 1 to E_2 with $\varepsilon = \frac{1}{3}$

$\exists E_3 \subseteq E_2 \subseteq E_1 \subseteq E$ with $\nu(E_3) \geq \nu(E_2) \geq \nu(E_1) \geq \nu(E)$ such that $\forall A \in \mathcal{F}$ $A \subseteq E_3$ then $\nu(A) \geq -\frac{1}{3}$

$\exists E_k \subseteq \ldots \subseteq E$ with $\nu(E_k) \geq \ldots \geq \nu(E)$ s.t. $\forall A \in \mathcal{F}$, $A \subseteq E_k \Rightarrow \nu(A) \geq -\frac{1}{k}$

Take $F = \bigcap_k E_k$ we have

$F \subseteq E$, $\quad \nu(F) = \lim_n \nu(E_\varepsilon) \geq \nu(E)$

and if $A \in \mathcal{F}$ $\quad F \cap A \subseteq F \subseteq E_k \quad \forall k$

then $\nu(F \cap A) \geq -\frac{1}{k} \quad \forall k$

$\Rightarrow \nu(F \cap A) \geq 0$ QED

proof (Hahn's decomposition) $\nu : \mathcal{F} \to [-\infty, +\infty[$

consider this case

I consider $\alpha = \sup \{ \nu(A),\ A \in \mathcal{F} \}$

mind the (for the moment I don't know that $\alpha < +\infty$)

I consider $(E_n)_n$ in $\mathcal{F}$

s.t. $\lim_n \nu(E_n) = \alpha$

it is not restrictive to suppose that $-\infty < \nu(E_n) < +\infty$

so we apply lemma 2 to E_n.

$\forall n,$ $\exists F_n \in \mathcal{F}$ with $F_n \subseteq E_n$, $\nu(F_n) \geq \nu(E_n)$

and $\forall A \in \mathcal{F}$ $\nu(A \cap F_n) \geq 0$.

$\lim_n \nu(F_n) = \alpha$

I set $G_n = \bigcup_{j=1}^{n} F_n$ so also $\lim_n \nu(G_n) = \alpha$

and $(G_n)_n$ is an increasing sequence

$G_n \subseteq G_{n+1}$

and G_n is non negative (Ex. union of non negative sets)

Put $P = \bigcup_n G_n = \bigcup_n F_n$

$\nu(P) = \alpha$

P is non negative

To conclude, it is sufficient to prove that $\Omega \setminus P = N$ is non positive

by contradiction, suppose N is not non positive.

$\neg($non positive : $\forall A \in \mathcal{F}$, $\nu(A \cap N) \leq 0)$

$\exists A \in \mathcal{F}$ s.t. $\nu(A \cap N) > 0$

$\Updownarrow$

$\boxed{\exists \tilde{A} \in \mathcal{F}, \quad \tilde{A} \subseteq N \text{ and } \nu(\tilde{A}) > 0}$

consider $\underbrace{P \cup \tilde{A}}_{\text{disjoint}}$ we know $\tilde{A} \subseteq N = \Omega \setminus P$

$\nu(P \cup \tilde{A}) = \underbrace{\nu(P)}_{=\alpha} + \nu(\tilde{A}) = \alpha + \underbrace{\nu(\tilde{A})}_{>0}$

$\boxed{\nu(P \cup \tilde{A}) > \alpha}$ impossible

sup

proof of uniqueness of the decomp.

suppose $(\tilde{P}, \tilde{N})$ are two dec.

(P,N) and then $\nu(P \setminus \tilde{P}) = \nu(P \cap \tilde{N})$

P non negative $\Rightarrow \nu(P \cap \tilde{N}) \geq 0$

$\tilde{N}$ non negative $\Rightarrow \nu(P \cap \tilde{N}) \leq 0$

$\Rightarrow \nu(P \setminus \tilde{P}) = 0$ QED.

Figure 9: Hans Hahn (1879–1934)

Proof. We suppose that, for all $E \in \mathcal{A}$, $-\infty \leq \nu(E) < +\infty$. We set $\alpha = \sup\{\nu(E)\,,\, E \in \mathcal{A}\}$. We take $(E_n)_n$ a sequence in $\mathcal{A}$ such that $\lim_n \nu(E_n) = \alpha$. It is not restrictive to suppose that, for all n, $-\infty < \nu(E_n) < +\infty$. We apply Lemma 8 to each set E_n, obtaining a sequence $(F_n)_n$ such that, for all n,

$$F_n \subseteq E_n,\ \nu(F_n) \geq \nu(E_n) \text{ and, for all } A \in \mathcal{A},\ \nu(A \cap F_n) \geq 0.$$

We set now $G_n = \cup_{j=1}^n F_j$. We have that $\nu(G_n) \geq \nu(F_n)$ since $G_n = \cup_{k=1}^n \tilde{F}_k$, where $\tilde{F}_n = F_n$ and $\tilde{F}_k = F_k \setminus (\cup_{j=k+1}^n F_j)$ for $k = 1, \ldots, n-1$. The sets $\tilde{F}_k$ are pairwise disjoint and have non negative measure. Remark that, for all n, G_n is a non negative set (in fact is the union of non negative sets) and the sequence $(G_n)_n$ is increasing. We set $P = \cup_{n=1}^{+\infty} F_n = \cup_{n=1}^{+\infty} G_n$. We have that

$$\nu(P) = \lim_n \nu(G_n) = \lim_n \nu(F_n) = \lim_n \nu(E_n) = \alpha \text{ and consequently } \alpha \in \mathbb{R}.$$

It is immediate to see that P is non negative, as it is union of non negative sets.

It remains to prove that $\Omega \setminus P$ is non positive. Suppose by contradiction there exists A contained in $\Omega \setminus P$ such that $\nu(A) > 0$. Then $\nu(P \cup A) = \alpha + \nu(A) > \alpha$, and this is impossible.

Suppose now that (P, N) and (P', N') are two Hahn's decomposition. Considering that $P \setminus P' = P \cup N'$, using the fact that P si non negative we have $\nu(P \setminus P') \geq 0$ and using the fact that N' is non positive we have $\nu(P \setminus P') \leq 0$ and the conclusion follows. The other cases are similar. □

Exercise 1. *Let ν be a signed measure on a measurable space $(\Omega, \mathcal{A})$. Suppose that for all $E \in \mathcal{A}$, $-\infty \leq \nu(E) < +\infty$, i.e. $\nu(\mathcal{A}) \in [-\infty, +\infty[$.*

Prove that if $\alpha = \sup\{\nu(E)\,,\, E \in \mathcal{A}\}$, then $\alpha < +\infty$.

5.3 Total variation of a measure

The content of this paragraph can be find in [2, §19].

Definition 14. *Let ν be a signed measure on $(\Omega, \mathcal{A})$. Let (P, N) a Hahn's decomposition. We set, for all $E \in \mathcal{A}$,*

$$\begin{aligned} \nu^+(E) &= \nu(E \cap P), \\ \nu^-(E) &= -\nu(E \cap N), \\ |\nu|(E) &= \nu^+(E) + \nu^-(E). \end{aligned}$$

ν^+, ν^- and $|\nu|$ are positive measures and they are called positive variation, negative variation and total variation of ν, respectively.

We give a characterization of the total variation of a measure.

Theorem 17. *Let ν be a signed measure on $(\Omega, \mathcal{A})$. We set, for all $E \in \mathcal{A}$,*

$$\mu(E) = \sup\{\sum_{j=1}^{k} |\nu(E_j)| \mid E = \cup_{j=1}^{k} E_j, \quad E_j \cap E_h = \emptyset \ \ \textit{if} \ \ j \neq h\}. \tag{9}$$

Then, for all $E \in \mathcal{A}$,

$$\mu(E) = |\nu|(E).$$

Proof. Let (P, N) a Hahn's decomposition of ν. We have that

$$|\nu(E_j)| = |\nu(E_j \cap P) + \nu(E_j \cap N)| \leq |\nu(E_j \cap P)| + |\nu(E_j \cap N)| = |\nu|(E_j).$$

Consequently

$$\sum_{j=1}^{k} |\nu(E_j)| \leq \sum_{j=1}^{k} |\nu|(E_j) = |\nu|(E),$$

and we obtain that

$$\mu(E) \leq |\nu|(E).$$

Converserly, if we write $E = (E \cap P) \cup (E \cap N)$, since $(E \cap P) \cap (E \cap N) = \emptyset$, we have

$$\mu(E) = \sup\{\sum_{j=1}^{k} |\nu(E_j)| \ldots\} \geq |\nu(E \cap P)| + |\nu(E \cap N)| = |\nu|(E).$$

□

Remark 9. *Suppose not having proved Hahn's decomposition theorem. It is still possible to prove that μ, defined by (9), is a positive measure. To prove that $\mu(\emptyset) = 0$ is immediate. To prove that μ is countably additive we proceed in the following way. Let $(A_n)_n$ be a sequence in $\mathcal{A}$ consisting of pairwise disjoint subsets of Ω. Let $A = \bigcup_n A_n$. Let*

$$\beta < \sup\{\sum_{j=1}^{k} |\nu(E_j)| \mid A = \cup_{j=1}^{k} E_j, \quad E_j \cap E_h = \emptyset \ \ \textit{if} \ \ j \neq h\} = \mu(A).$$

Total variation of a measure

def. Let $(\Omega, \mathcal{F})$ measurable space. Let ν a signed measure
Let (P, N) a Hahn's decomposition

We define
$\nu^+(A) = \nu(A \cap P)$ positive variation of ν
$\nu^-(A) = -\nu(A \cap N)$ negative " of ν
$|\nu|(A) = \nu^+(A) + \nu^-(A)$ total variation of ν

we have that $\nu^+, \nu^-, |\nu|$ are positive measures
and
$$\boxed{|\nu(A)| = |\nu^+(A) - \nu^-(A)|} \quad \forall A \in \mathcal{F}$$

Theorem Let $(\Omega, \mathcal{F}, \nu)$ as before

Define $\mu : \mathcal{F} \to [0, +\infty[$
in the following way

$$\boxed{\mu(A) = \sup \left\{ \sum_{j=1}^{m} |\nu(E_j)| \right\}}$$

made on the set of all the possible decomposition of A
$A = E_1 \cup E_2 \cup \dots \cup E_m$ finite number of set in $\mathcal{F}$
with $E_j \cap E_k = \emptyset$ if $j \neq k$

Then $\mu = |\nu|$

proof.

Let $A \in \mathcal{F}$
Let $A = E_1 \cup E_2 \cup \dots \cup E_k$ with $E_j \in \mathcal{F}$ pairwise disjoint.

then for each E_j

we know
$$\nu(E_j) = \nu^+(E_j) - \nu^-(E_j)$$
$$|\nu(E_j)| = |\nu^+(E_j) - \nu^-(E_j)| \leq |\nu^+(E_j)| + |\nu^-(E_j)| \leq \nu^+(E_j) + \nu^-(E_j) = |\nu|(E_j)$$

so $\sum_j |\nu(E_j)| \leq \sum_j |\nu|(E_j) = |\nu|(A)$

this is true for all the decomp. of A

so that $\mu(A) \leq |\nu|(A)$

conversely take $A \in \mathcal{F}$

we have $A = (A \cap P) \cup (A \cap N)$ (P, N) Hahn's decom.

so that

$$|\nu|(A) = \underbrace{|\nu(A \cap P)|}_{= \nu^+(A)} + \underbrace{|\nu(A \cap N)|}_{= \nu^-(A)} \leq \sup\{\Sigma\} = \mu(A)$$

$$|\nu|(A) \leq \mu(A)$$

QED

Remark suppose not having proved Hahn's theorem
one can still define

$$\mu(A) = \sup \left\{ \sum_{j=1}^{m} |\nu(E_j)| : A = \bigcup_j E_j \text{ finite pairwise disjoint} \right\}$$

μ is a function from $\mathcal{A}$ to $[0,+\infty[$

it is possible to prove that μ is positive measure

proving i) $\mu(\emptyset) = 0$ immediate

ii) μ is countably additive

so that μ is a measure (this is for free if one does Hahn's decomposition)

this in the note as exercise

Remark let now ν be a complex measure.

we know that $\mathrm{Re}\,\nu$, $\mathrm{Im}\,\nu$ are signed measures

Any how, how to define the total variation of ν ?

through Hahn's it is possible to define the total variation of $\mathrm{Re}\,\nu$ and $\mathrm{Im}\,\nu$

on the contrary

ν complex

$$|\nu|(A) = \sup \left\{ \sum_{j=1}^{m} |\nu(E_j)| : A = \bigcup_{j=1}^{m} E_j \text{ finite pairwise disjoint} \right\}$$

Theorem (book of Rudin)

in this case (ν complex measure)
$|\nu|$ is a positive finite measure

as the exercise before

Rudin

Remark consider ν a complex measure

then $\nu = (\mathrm{Re}\,\nu)^+ - (\mathrm{Re}\,\nu)^- + i\left((\mathrm{Im}\,\nu)^+ - (\mathrm{Im}\,\nu)^-\right)$

Jordan's decomposition of a complex measure

$= \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$

with $\nu_1, \nu_2, \nu_3, \nu_4$ finite positive measures