GAUGE INVARIANCE

THE HAMILTONIAN OF A CHARGED PARTICLE DEPENDS ON A VECTOR POTENTIAL $\bar{A}$

(55) $$\hat{H} = \frac{1}{2m}\left[\hat{p} - q\bar{A}(\bar{r},t)\right]^2 + Q\varphi(\bar{r},t)$$

SINCE $\bar{A}$ IS DEFINED ONLY UP TO SOME GAUGE CHOICE THE WAVEFUNCTION IS NOT A GAUGE INVARIANT OBJECT.

TO EXPLORE THE GAUGE FREEDOM WE START CONSIDERING

$$\bar{A} \to \bar{A}' = \bar{A} + \bar{\nabla}\lambda \quad ; \quad \varphi \to \varphi' = \varphi - \partial_t \lambda$$

WHERE $\lambda(\bar{r},t)$ DENOTES ARBITRARY CONTINUUM AND DERIVABLE SCALAR FUNCTION. UNDER THE GAUGE TRANSFORMATION

$$i\hbar\partial_t \psi = \hat{H}[\bar{A}]\psi \to i\hbar\partial_t \psi' = H[\bar{A}]\psi'$$

WHERE THE WAVEFUNCTION ACQUIRES ADDITIONAL PHASE.

(56) $$\boxed{\psi'(\bar{r},t) = \exp\left[i\frac{Q}{\hbar}\lambda(\bar{r},t)\right]\psi(\bar{r},t)}$$

BUT THE PROBABILITY DENSITY IS CONSERVED $|\psi'(\bar{r},t)|^2 = |\psi(\bar{r},t)|^2$ (57)

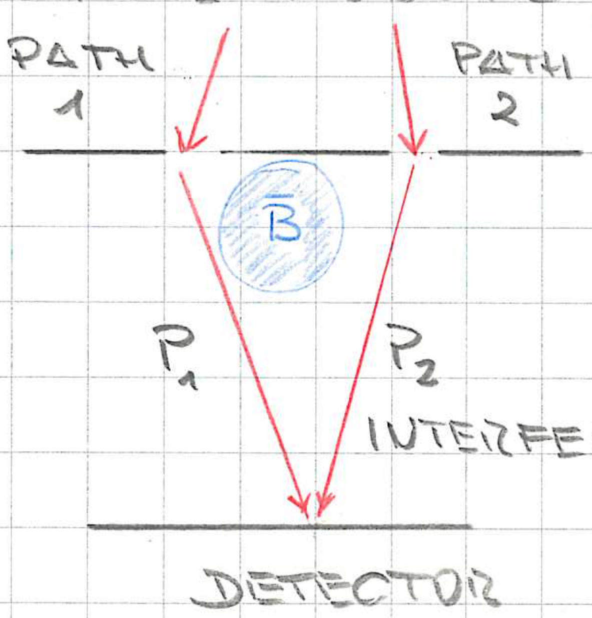
THEREFORE, IF $i\hbar\partial_t\psi = \hat{H}[\bar{A}]\psi$ WE HAVE

$i\hbar\partial_t\psi' = \hat{H}[\bar{A}']\psi'$ (58)

PHYSICAL CONSEQUENCES

LET'S CONSIDER (56) AND A CHARGE Q TRAVELLING ALONG A PATH, P, IN

WHICH THE MAGNETIC FIELD $\bar{B}=0$ ALONG THE PATH BUT IN THE, S, AREA ENCLOSED WITHIN THE PATHS $\bar{B}\neq 0$.



HOWEVER, WHERE $\bar{B}=0 \not\Rightarrow \bar{A}=0$: ANY $\lambda(\bar{r})$ SUCH THAT $\bar{A}=\bar{\nabla}\lambda$ LEADS $\bar{B}=0$. IN TRAVERSING A PATH THE WAVEFUNCTION AQUIRES A PHASE $\phi=\frac{Q}{\hbar}\lambda \Rightarrow d\phi=\frac{Q}{\hbar}d\lambda$

$$\bar{A}=\frac{d\lambda}{dx} \text{ (IN 1D)} \Rightarrow d\phi=\frac{Q}{\hbar}\bar{A}\cdot dx \rightarrow$$

$$\phi=\frac{Q}{\hbar}\int_P \bar{A}\cdot d\bar{r} \text{ (IN 3D)}. \quad (59)$$

IF WE CONSIDER P_1 AND P_2 WITH THE SAME INITIAL AND FINAL POINTS (SEE FIGURE) THE RELATIVE PHASE OF THE WAVEFUNCTIONS IS

$$\Delta\phi=\frac{Q}{\hbar}\int_{P_1}\bar{A}\cdot d\bar{r}-\frac{Q}{\hbar}\int_{P_2}\bar{A}\cdot d\bar{r}= \quad (60)$$

$$=\frac{Q}{\hbar}\oint \bar{A}\cdot d\bar{r} \xRightarrow{\text{STOKES}} \frac{Q}{\hbar}\int_S \bar{B}\cdot ds$$

WHERE $\int_A$ RUNS OVER THE AREA ENCLOSED BY THE LOOP FORMED BY P_1 AND P_2. THE RELATION $\Delta\phi = \frac{Q}{\hbar}\int_{S'} \bar{B}\cdot d\bar{S}$ $\Rightarrow$

FOR THE PATHS P_1 AND P_2 THE WAVEFUNCTION COMPONENTS ACQUIRE A RELATIVE PHASE DIFFERENCE

$$\Delta\phi = \frac{Q}{\hbar} \times \text{FLUX OF } \bar{B}_{S'} \quad (60)$$

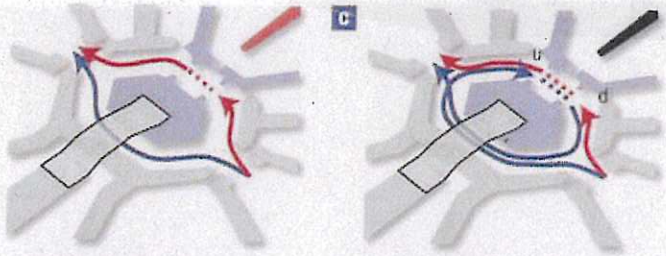
$\Rightarrow$ IF $\phi_{S'}(\bar{B}) \neq 0$, EVEN IF $\bar{B} = 0$ ON THE PATHS P_1 AND P_2, $\psi(\bar{r})$ GAINS A NON-ZERO RELATIVE PHASE. ON THE DETECTOR WE WILL DETECT THE INTERFERENCE BETWEEN

$$\left|\psi_{P_1}(\bar{r},t) + \psi_{P_2}(\bar{r},t)\right|^2 \quad (61)$$

WHICH CAN BE CONSTRUCTIVE OR DISTRUCTIVE. THIS PHENOMENON IS KNOWN AS THE AHARANOV-BOHM EFFECT AND IT LEADS TO QUANTUM INTERFERENCE BETWEEN MASSIVE CHARGED PARTICLES MOVING IN A SOLID (SEMICONDUCTOR) AND IT CAN INFLUENCE OBSERVABLE PROPERTIES.

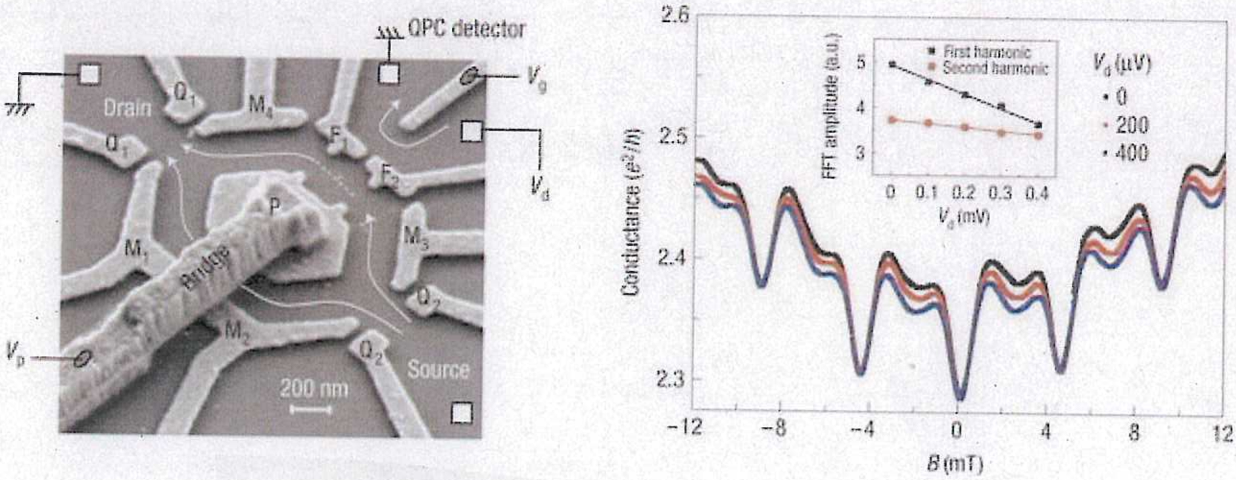
- OBSERVATION THE SIMILARITY OF THIS MECHANISM OF INTERFERENCE WITH THE CLASSICAL E.M. WAVE INTERFERENCE IN THE YOUNG INTERFEROMETER IS

CLEAR. FOR THE AHARANOV-BOHM THE e^- (OR ANY CHARGE PARTICLE) ARE TREATED AS A WAVE $\Rightarrow$ $\Delta\varphi = 2\pi n$ (WHERE n IS 0, 1, 2, … EXPECT CONSTRUCTIVE INTERFERENCE $\Rightarrow$ IF WE VARY $\bar{B}$ WE EXPECT OSCILLATIONS OF THE CONDUCTANCE



First harmonic (h/e oscillation) path information

Second harmonic ($h/2e$ oscillation) no path information



- <u>SUMMARY</u> CHARGE PARTICLE IN A FIELD.

1. START FROM CLASSICAL LAGRANGIAN FOR A PARTICLE MOVING IN A STATIC E.M. FIELD

$$L = \frac{1}{2} m v^2 - Q\varphi + Q\bar{v}\cdot\bar{A} \Rightarrow$$

$$H = \frac{1}{2m}\left(\bar{p} - Q\bar{A}(\bar{r},t)\right)^2 + Q\varphi(\bar{r},t) \Rightarrow$$

FOR $\bar{\nabla}\cdot\bar{A}=0$ AND INTRODUCING THE $\hat{H}$, $\hat{p}$, $\hat{x}$ OPERATORS ($\Leftrightarrow$ TRANSFORMING THE CLASSICAL H FOR A CHARGE MOVING INTO AN E.M. FIELD) $\Rightarrow$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \frac{iQ\hbar}{m}\bar{A}\cdot\bar{\nabla} + \frac{e^2}{2m}A^2 + Q\varphi$$

- THIS $\hat{H}$ APPLIED TO ATOMIC HYDROGEN IN A STATIONARY UNIFORM $\bar{B}$ ($\bar{B}=B\hat{z}$) LEADS TO

$$\hat{H} = \frac{1}{2m}\left[\hat{p}_r^2 + \frac{\hat{L}^2}{r^2} + eB\hat{L}_z + \frac{1}{4}e^2B^2(x^2+y^2)\right] + V(\bar{r})$$

WHERE $V(\bar{r}) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$

- FOR WEAK FIELDS THE DIAMAGNETIC TERM IS NEGLIGIBLE $\Rightarrow$ IGNORING THE e^- SPIN THE $\bar{B}$ SPLIT THE ORBITAL DEGENERACY LEADING TO THE NORMAL ZEEMAN EFFECT

$$E_{n\ell m} = -\frac{1}{n^2}R_y + \mu_B B m$$

- IF THE DIAMAGNETIC TERMS COMPETES WITH COULOMB ENERGY SCALE ($-R_y/n^2$) THE SYSTEM ENTERS IN A QUANTUM CHAOTIC REGIME. (NOT STUDIED HERE)
- GAUGE INVARIANCE OF THE E.M. FIELDS $\Rightarrow$ THE WAVEFUNCTION IS NOT GAUGE

INVARIANT.

- UNDER THE GAUGE TRANSFORMATION $\bar{A} \to \bar{A}' = \bar{A} + \bar{\nabla}\lambda$; $\varphi \to \varphi' = \varphi - \partial_t \lambda$ THE WAVE FUNCTION ACQUIRES ADDITIONAL PHASE $\psi'(\bar{r},t) = \exp\left[i\frac{q}{\hbar}\lambda(\bar{r},t)\right]\psi(\bar{r},t)$

- $\Rightarrow$ AHARANOV-BOHM EFFECT: CHARGES ENCIRCLING A MAGNETIC FLUX GAIN A RELATIVE PHASE $\Delta\phi = \frac{q}{\hbar}\int_{S'} \bar{B}\cdot d\bar{s}$

- FREE ELECTRONS IN A MAGNETIC FIELD: LANDAU LEVELS.

WE NOW DISCUSS WHAT HAPPENS WHEN FREE ELECTRONS EXPERIENCE A MAGNETIC FIELD WHICH INFLUENCES THE ORBITAL MOTION? WE HAVE TO FACE TWO CASES e^- IN A VACUUM (ACCELERATORS - PLASMA CLOUD) AND IN A SOLID (METAL) NOW WE TREAT THIS LAST CASE

$$\hat{H} = \frac{1}{2m}\left(\hat{p} - q\bar{A}(\bar{r},t)\right)^2$$

$q = -e$

BEING $\varphi(\bar{r},t) = 0$

IN THIS CASE CLASSICAL ORBITS CAN BE MACROSCOPIC AND THERE IS NO REASON TO NEGLECT THE DIAMAGNETIC CONTRIBUTION.
WE START TO ANALYSE THE PROBLEM FROM THE CLASSICAL PHYSICS. THE LORENTZ FORCE CAUSES THE ELECTRON TO SPIRAL AROUND

THE CENTRIFUGA FORCE MUST BALANCE THE LORENTZ FORCE $\frac{mv^2}{r} = |e| vB$ (FOR EXECISE

WE DO THAT IN GAUSS UNITS $\Rightarrow \frac{mv^2}{r} = \frac{|e|}{c} Bv$

$\Rightarrow r = \frac{mcv}{|eB|}$ WHICH IS THE LARMOR RADIUS (OR CYCLOTRON RADIUS) $\Rightarrow$

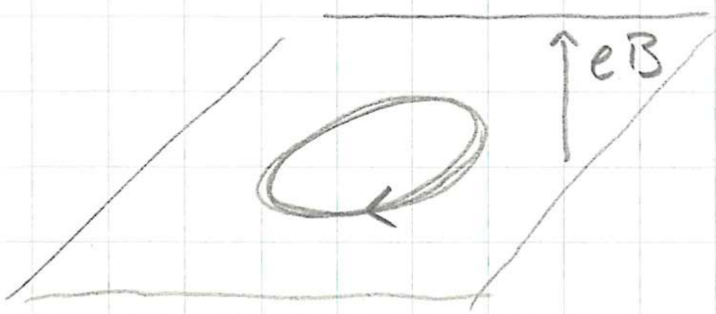
(62) $\omega = 2\pi \frac{v}{2\pi r} = \frac{|eB|}{mc} \Rightarrow$ DOES NOT DEPEND

ON THE CYCLOTRON RADIUS AND IT GIVES A CHARACTERISTIC TIME-SCALE OF THE PROBLEM

THE ENERGY IS GIVEN $E = \frac{m}{2} v^2 = \frac{m}{2} r^2 \omega^2 \Rightarrow$ (63)

$E \propto r^2$ FOR A FIXED $\bar{B}$.

FOR DEFINITENESS, WE ASSUME $eB > 0 \Rightarrow$ $\bar{B}$ POINTS DOWNWARD $/\!/ \hat{z} \Rightarrow B < 0$ FOR $Q = -e < 0 \Rightarrow$ THE MOTION IS CLOCKWISE.



- OBSERVATION EVEN THOUGH THE SYSTEM IS BOTH ROTATIONALLY (AROUD $\hat{z}$) AND TRANSLATIONALLY INVARIANT, IT IS CURIOUS THAT A VECTOR POTENTIAL THAT IS INVARIANT UNDER BOTH OF THEM CAN NOT BE FOUND.

ONE COMMON CHOICE PRESERVES THE ROTATIONAL INVARIANCE (THE SYMMERIC GAUGE)

$$\bar{A} = (A_x, A_y) = \frac{B}{2}(-y, x) \quad \text{WHILE} \qquad (64)$$

$\bar{A} = (A_x, A_y) = B(-y, 0)$ AND YET ANOTHER PRESERVES THAT ALONG THE Y-AXIS, $\bar{A} = (A_x, A_y) = B(0, x)$. THE CLASSICAL H

$$H = \frac{1}{2m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)^2$$

• **OBSERVATION** SOMETIMES $p - \frac{e}{c}\bar{A}$ IS SHORTEN IN $\Pi \Rightarrow \left(p - \frac{e}{c}\bar{A}\right)^2 = \Pi^2$.

THE HAMILTONIAN EQ OF MOTION IS THEN

$$\dot{x} = \partial_{\bar{p}} H = \frac{1}{m}\left(\bar{p} - \frac{e}{c}\bar{A}\right); \quad \dot{\bar{p}} = -\partial_x H =$$

$$= \frac{e}{mc}\left(p_i - \frac{e}{c}A_i\right)\bar{\nabla} A_i \Rightarrow \qquad (65)$$

$$\ddot{x} = \frac{1}{m}\left(\dot{\bar{p}} - \frac{e}{c}x_i \bar{\nabla}_i A\right) =$$

$$= \frac{1}{m}\left[\frac{e}{mc}\left(p_i - \frac{e}{c}A_i\right)\bar{\nabla} A_i - \frac{e}{c}\frac{1}{m}\left(p_i - \frac{e}{c}A_i\right)\bar{\nabla}_i \bar{A}\right] =$$

$$= \frac{e}{mc}\left(v_i \bar{\nabla} A_i - v_i \bar{\nabla}_i A\right) = \frac{e}{mc}\left(\bar{v} \times \bar{B}\right)$$

THIS IS THE NEWTON LAW WITH THE LORENTZ FORCE $m\ddot{\bar{x}} = \frac{e}{c}\left(\bar{v} \times \bar{B}\right)$ IN ITS COMPONENTS

$$\bar{X} \equiv (x, y, z) \Rightarrow \ddot{x} = \frac{eB}{mc}v_y = \omega v_y$$

$$\ddot{y} = -\frac{eB}{mc}v_x = -\omega v_x \qquad (66)$$

THEREFORE A GENERAL SOLUTION IS

$$x = X + r\cos\omega t$$
$$y = Y - r\sin\omega t$$