Remarks on the Nullstellensatz

1) $X \subseteq \mathbb{A}^n_K$ hypersurface $X = V(F)$, $F = F_1^{r_1} F_2^{r_2} \cdots F_s^{r_s}$ decomposition in irreducible factors, then $X = V(F_1 F_2 \cdots F_s)$

$F' = F_1 F_2 \cdots F_s$ reduced equation of X

It $\in$ $\sqrt{\langle F \rangle} = \langle F' \rangle$: if $H^r = AF$, consider the factorization in irreducible factors of both members $\Longrightarrow$ every $F_i \mid H^r \Longrightarrow$ $F_i \mid H$, therefore $H \in \langle F' \rangle$. $X = V(F')$

If K is algebraically closed, then X determines its reduced equation: if $V(F) = V(G)$, with F, G square-free, then $I(V(F)) = I(V(G)) \Longrightarrow \langle F \rangle = \langle G \rangle$

$$\underset{\sqrt{F}}{\underset{\shortparallel}{I(V(F))}} = \underset{\sqrt{G}}{\underset{\shortparallel}{I(V(G))}}$$

so F, G coincide up to a constant. $x_1^2 + \cdots + x_n^2$

Similarly in the projective case.

2) If K is not algebraically closed, there are characterizations of ideals without zeros and of $I(V(\mathfrak{a}))$ depending on the field.

If $K=\mathbb{R}$ Positivstellensatz:

$I(V(\mathfrak{a})) = {}^{\mathbb{R}}\sqrt{\mathfrak{a}}$ real radical of $\mathfrak{a}$ $\quad \mathbb{R}[x_1, \dots, x_n]$

$${}^{\mathbb{R}}\sqrt{\mathfrak{a}} = \{F \in \mathbb{R}[x_1, \dots, x_n] \mid -F^{2m} = \sum H_i^2 + G,\ G \in \mathfrak{a},\ m > 0\}$$

This holds for "real closed" fields.

Projective closure of an affine algebraic set.

$$\mathbb{A}^n_K \cong U_0 \subseteq \mathbb{P}^n_K \qquad U_0 = \{[x_0, \dots, x_n] \mid x_0 \neq 0\}$$

$$j_0 : \mathbb{A}^n \longrightarrow U_0 \qquad\qquad \varphi_0 : U_0 \longrightarrow \mathbb{A}^n$$

$$(x_1, \dots, x_n) \longrightarrow [1, x_1, \dots, x_n] \qquad [y_0, \dots, y_n] \to \left(\frac{y_1}{y_0}, \dots, \frac{y_n}{y_0}\right)$$

$y_0 \neq 0$

$P(x_1, \dots, x_n)$ affine of P $\qquad [y_0, \dots, y_n]$ homogeneous coord. of P

$$\frac{y_1}{y_0} = x_1, \dots, \frac{y_n}{y_0} = x_n$$

$$\boxed{\begin{matrix} X \subseteq \mathbb{A}^n & \rightsquigarrow & Y \subseteq \mathbb{P}^n \\ \text{closed} & & \text{closed} \end{matrix}}$$ s.t. $\underline{Y \cap \mathbb{A}^n = X}$

$V(\alpha) = X$ $\qquad Y = V_p({}^h\alpha)$ $\qquad {}^h\alpha = \langle {}^hF \mid F \in \alpha \rangle$

$Y \subseteq \mathbb{P}^n$ closed $\qquad Y \cap \mathbb{A}^n$ closed

$Y = V_p(\beta)$ $\qquad Y \cap \mathbb{A}^n = V({}^a\beta)$

$X \subseteq \mathbb{A}^n$ closed $\qquad \overline{X} \subseteq \mathbb{P}^n$ closure of $X \subseteq U_0$ in the Zariski top. of $\mathbb{P}^n$

$\overline{X}$ is the minimum closed subset of $\mathbb{P}^n$ s.t.

$\overline{X} \cap \mathbb{A}^n = X$ $\qquad \overline{X}$ projective closure of X

$$\overline{X} = \underbrace{(\overline{X} \cap \mathbb{A}^n)}_{X} \cup \underbrace{(\overline{X} \cap H_0)}_{\text{points at infinity of } X} \qquad H_0 = \mathbb{P}^n \setminus U_0$$

$x_0 = 0$

$\overline{\mathbb{A}^n_K} = \overline{U_0} \subseteq \mathbb{P}^n_K$ Ass. K is infinite $\Longrightarrow$

$\overline{\mathbb{A}^n_K} = \mathbb{P}^n_K$ $\quad$ $\mathbb{A}^n_K$ is dense in $\mathbb{P}^n_K$

Pf. Consider $F(x_0, \dots, x_n)$ homog. of deg d s.t. $\underline{V_P(F) \supseteq \mathbb{A}^n_K}$: we want to prove that F vanishes on the whole $\mathbb{P}^n_K$.

$F(1, a_1, \dots, a_n) = 0 \quad \forall (a_1, \dots, a_n) \in \mathbb{A}^n_K$

$\quad {}^aF(a_1, \dots, a_n) \Longrightarrow \underline{{}^aF \in I(\mathbb{A}^n)} = (0)$

Claim $I(\mathbb{A}^n_K) = (0)$ $\quad$ Induction on n

$n = 1$ $\quad F(x) \in I(\mathbb{A}^1_K)$, K infinite $\Rightarrow F = 0$

$n \geq 2$ $\quad F(x_1, \dots, x_n) \in I(\mathbb{A}^n_K)$ $\quad \forall (a_1, \dots, a_n) \in K^n$

$F(a_1, \dots, a_n) = 0$. $\quad F(\underline{a_1, \dots, a_{n-1}}, x_n) \in K[x_n]$: vanishes $\forall a_n \in K$.

If $\exists a_1, \dots, a_{n-1}$ st. $F(a_1, \dots, a_{n-1}, x_n)$ has positive deg

$\Rightarrow$ it has fin. many zeros in K ; contradict.

$\forall (a_1, \dots, a_{n-1}) \in K^{n-1}$ $\quad F(a_1, \dots, a_{n-1}, x_n)$ has deg 0 in x_n

$\Rightarrow$ in $F(x_1, \dots, x_n)$ x_n does not occur

$F \in K[x_1, \dots, x_{n-1}]$ $\quad F \in I(\mathbb{A}^{n-1}_K) \underset{\text{induction}}{=} (0)$

$F = 0 = \underline{0 + 0} - \cdot + 0$

${}^aF = 0 \Rightarrow F = 0$

$\mathbb{A}^n_K$ is dense in $\mathbb{P}^n_K$

Prop: $X \subseteq \mathbb{A}^n_K$ affine algebraic variety

$\overline{X}$ $\qquad I_h(\overline{X}) = {}^hI(X) = \underline{\langle {}^hF \mid F \in I(X) \rangle}$

Pf: $I_h(\overline{X}) \subseteq {}^hI(X)$ $\qquad$ F homog. $\quad F \in I_h(\overline{X})$

$\Rightarrow \forall\, P \in X \quad F(P) = F(1, a_1, \ldots, a_n) = 0 = {}^aF(a_1, \ldots, a_n)$

$P = (a_1, \ldots, a_n)$

$\Rightarrow {}^aF \in I(X) \;\therefore\; {}^h({}^aF) \in {}^hI(X)$

$F = x_0^r \, \underbrace{{}^h({}^aF)}_{{}^hI(X)} \in {}^hI(X)$

${}^hI(X) \subseteq I_h(\overline{X}) \;\therefore\; G \in I(X), \quad P(a_1, \ldots, a_n) \in X$

$G(a_1, \ldots, a_n) = 0 \qquad {}^hG = x_0^d\, G\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$

${}^hG(1, a_1, \ldots, a_n) = G(a_1, \ldots, a_n) = 0$

${}^hG \in I_h(X), \quad X \subseteq \mathbb{P}^n$

To conclude: need to show $I_h(X) = I_h(\overline{X})$.

$$I_h(\overline{X}) = {}^hI(X)\langle {}^hF \mid F\in I(X)\rangle$$

X hypersurface $\quad X=V(F) \quad F$ reduced equation of X

<u>K alg. closed</u>. $\quad I(X)=(F) \qquad I_h(\overline{X})={}^h(F)$

$$(F)=\{GF \mid G\in K[x_1\ldots x_n]\}$$

$${}^h(F) = \langle {}^h(GF)={}^hG\,{}^hF \rangle = \langle {}^hF\rangle$$
$$G\in K[x_1\ldots x_n]$$

$$\overline{X}\cap H_0 = V_p({}^hF)\cap V_p(x_0)$$

$X \qquad I(X)=\langle F_1,\ldots,F_r\rangle$

$$I_h(\overline{X}) \overset{?}{=} \langle {}^hF_1,\ldots,{}^hF_r\rangle \quad \text{NO}$$

<u>Example</u> The skew cubic or twisted cubic

K infinite field

<u>Def</u>: affine skew cubic is the image of

$\varphi : \mathbb{A}^1_K \longrightarrow \mathbb{A}^3_K \qquad t \longrightarrow (t, t^2, t^3) \qquad x, y, z$

φ is injective, $\operatorname{Im}\varphi$ is closed in $\mathbb{A}^3$; X

$= \boxed{V(y - x^2, z - x^3)}$

$\varphi^{-1} : X \longrightarrow \mathbb{A}^1 \qquad (x, y, z) \longrightarrow x$

φ, φ^{-1} are continuous: X is homeom. to $\mathbb{A}^1_K$

$\alpha = \langle y - x^2, z - x^3 \rangle \qquad$ Is $\alpha = I(X)$?

$F \in I(X)$: $\forall t \in K \quad F(t, t^2, t^3) = 0$

$F \in K[x, y, z]$. Consider Taylor developm. in

(x, x^2, x^3): $\quad F = F(x, x^2, x^3) + \frac{\partial F}{\partial y}(x, x^2, x^3)(y - x^2) + \frac{\partial F}{\partial z}(x, x^2, x^3)(z - x^3) + \cdots$

F vanishes at $(x, x^2, x^3) \Longleftrightarrow F \in \langle y - x^2, z - x^3 \rangle$

$I(X) = \langle y - x^2, z - x^3 \rangle$

$\rho : K[x,y,z] \longrightarrow K[t]$ surjective

$F(x,y,z) \longrightarrow F(t,t^2,t^3)$ $\quad \operatorname{Ker}\rho = I(X) = \langle y-x^2, z-x^3 \rangle$

$\dfrac{K[x,y,z]}{\langle y-x^2, z-x^3\rangle} \simeq K[t]$ integral domain $\Longrightarrow$

α is a prime ideal $\Longrightarrow$

$\alpha = \sqrt{\alpha}$

$\overline{X}$ $\qquad I_h(\overline{X}) = {}^h\alpha \neq \langle x_0x_2 - x_1^2,\ x_0^2x_3 - x_1^3 \rangle$

$\qquad\qquad\qquad\qquad\qquad\qquad \langle {}^h(y-x^2),\ {}^h(z-x^3) \rangle$

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{\varphi} & X \subseteq \mathbb{A}^3 \\ \downarrow j_0 & & \downarrow \\ \mathbb{P}^1 & \overset{\psi}{\dashrightarrow} & \mathbb{P}^3 \end{array} \qquad \begin{array}{ccc} t & \xrightarrow{\varphi} & (t,t^2,t^3) \\ \downarrow & & \downarrow \\ [1,t] & \longrightarrow & [1,t,t^2,t^3] \end{array}$$

$\underline{\text{naturally}}$

$[x_0, x_1, x_2, x_3]$

$[\lambda,\mu]$

$[\lambda,\mu] \overset{\psi\,?}{\dashrightarrow} \left[1, \frac{\mu}{\lambda}, \left(\frac{\mu}{\lambda}\right)^2, \left(\frac{\mu}{\lambda}\right)^3\right] = \left[1, \frac{\mu}{\lambda}, \frac{\mu^2}{\lambda^2}, \frac{\mu^3}{\lambda^3}\right] =$

$\lambda \neq 0$

$\left[1, \frac{\mu}{\lambda}\right] \qquad\qquad = [\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$

$\underline{\text{Def}}.\ \psi([\lambda,\mu]) = [\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$

$\mathbb{P}^1 = \mathbb{A}^1 \cup \{[0,1]\}$ $\qquad \psi|_{\mathbb{A}^1} = \varphi$, $\quad \psi([0,1]) = \underline{[0,0,0,1]}$

$\psi(\mathbb{P}^1) = \varphi(\mathbb{A}^1) \cup \varphi([0,1]) = X \cup \{[0,0,0,1]\}$

Claim $\overline{X} = X \cup \{[0,0,0,1]\}$

1) Any closed subset of $\mathbb{P}^3$ containing X, contains also $[0,0,0,1]$ $\Longrightarrow$ $X \cup \{[0,0,0,1]\} \subseteq \overline{X}$

G homog. pol. vanishing on X:
$G(1,t,t^2,t^3) = 0 \quad \forall t \in K$
$\Downarrow$
$G(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3) = 0 \quad \forall \lambda \neq 0, \forall \mu \in K$

K infinite: as pol. in λ, $\forall \mu$ infinitely many zeros
$\Rightarrow G(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3) = 0 \quad \forall \lambda, \mu \in K$

2) $\boxed{X \cup \{[0,0,0,1]\} \text{ is closed in } \mathbb{P}^3}$ $\varphi(\mathbb{P}^1)$ closed in $\mathbb{P}^3$

We look for a homog. ideal I s.t. $V_P(I) = X \cup \{[0001]\}$

$$(*)\quad \begin{cases} x_0 = \lambda^3 \\ x_1 = \lambda^2\mu \\ x_2 = \lambda\mu^2 \\ x_3 = \mu^3 \end{cases}$$

We have to eliminate λ, μ from these parametric equations $\varphi(\mathbb{P}^1)$

$$\begin{cases} x_0x_3 = x_1x_2 \\ x_0x_2 = x_1^2 \\ x_1x_3 = x_2^2 \end{cases} \qquad \varphi(\mathbb{P}^1) \subseteq V_P(x_0x_3 - x_1x_2,\ x_0x_2 - x_1^2,\ x_1x_3 - x_2^2)$$

We want to prove that $=$

$P[y_0, \ldots, y_3]$ s.t. $y_0 y_3 = y_1 y_2$, $y_0 y_2 = y_1^2$, $y_1 y_3 = y_2^2$

λ, μ s.t. $P[\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$?

$y_0 \neq 0$ or $y_3 \neq 0$

$y_0 \neq 0$ $\quad P[y_0^3, y_0^2 y_1, y_0^2 y_2, y_0^2 y_3]$

$\boxed{\lambda = y_0, \ \mu = y_1}$ $\quad \lambda^3 \quad \lambda^2\mu \quad \lambda\mu^2 = y_0 y_1^2 \quad \mu^3 = y_1^3$

$y_0 y_2 = y_1^2$

$y_0^2 y_2 = y_0 y_1^2$

$y_1^3 = y_1 (y_0 y_2) = y_0 (y_0 y_3) = y_0^2 y_3$

If $y_0 \neq 0$ $\quad P = \varphi([y_0, y_1])$

$y_3 \neq 0$ $\quad P[\underbrace{y_0 y_3^2, y_1 y_3^2}, y_2 y_3^2, y_3^3]$

$\mu = y_3$ $\quad y_2 y_3^2 = \lambda\mu^2, \quad y_3^3 = \mu^3$

$\lambda = y_2$ $\quad \varphi[y_2, y_3]$

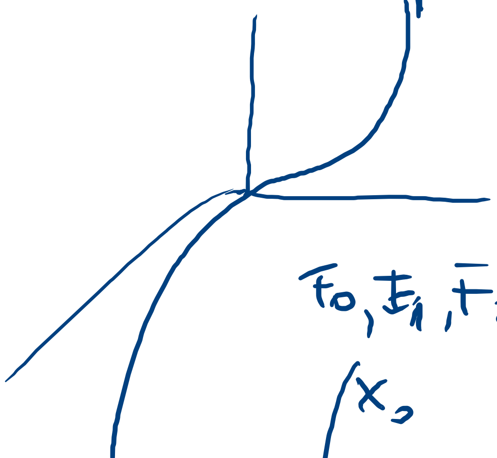
$\psi(\mathbb{P}^1) = \overline{X}$

$$\underbrace{V_p\langle x_0x_2 - x_1^2,\ x_0^2x_3 - x_1^3\rangle}_{\cap\, V_p(x_0)} \neq \overline{X}$$

$$\begin{cases} x_0x_2 - x_1^2 = 0 \\ x_0^2x_3 - x_1^3 = 0 \\ x_0 = 0 \end{cases}$$

$V_p(x_0, x_1) \subset \mathbb{P}^3$ a projective line

$\overline{X} \cap V_p(x_0) = \{[0,0,0,1]\}$



$$\langle \overset{F_1}{x_0x_3 - x_1x_2},\ \overset{F_2}{x_0x_2 - x_1^2},\ \overset{F_0}{x_1x_3 - x_2^2}\rangle = p$$

$V_p(p) = \overline{X}$

F_0, F_1, F_2 are the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \in M(2\times 3, K[x_0,x_1,x_2,x_3])$$

Claim $I_a(\overline{X}) = p$