$X \subseteq \mathbb{A}^3$ $\qquad \overline{X} \subseteq \mathbb{P}^3$

$\qquad\qquad \overline{X} = X \cup \{[0,0,0,1]\} = V_p(F_0, F_1, F_2)$

$$\begin{cases} x_0 = \lambda^3 \\ x_1 = \lambda^2\mu \\ x_2 = \lambda\mu^2 \\ x_3 = \mu^3 \end{cases}$$

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

$I_h(\overline{X}) = \langle F_0, F_1, F_2 \rangle$

Rmk $I_h(\overline{X})$ is homogeneous $=$ $\langle F_0, F_1, F_2 \rangle$"

To prove they are equal, it is enough to prove that $\forall d \geq 0$ $\underline{I_h(\overline{X})_d} = \underline{\langle F_0, F_1, F_2 \rangle_d}$

$K[x_0, \dots, x_3]_d$ $\quad$ K-vector space gen. by the monomials of deg d

$\dim = \binom{4+d-1}{d} = \binom{d+3}{d} = \underline{\binom{d+3}{3}}$

we try to compare their dim as K-vector spaces

$\rho_d : R_d \longrightarrow K[\lambda,\mu]_{3d}$ K-linear map, surjective

$R = K[x_0, \dots, x_3]$

$$\underset{d}{F(x_0 \dots x_3)} \longrightarrow \underset{3d}{F(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3)}$$

$\mathrm{Ker}\,\rho_d = \{F \text{ homog. of deg } d \mid F(\lambda^3, \dots, \mu^3) = 0\} =$

$= I_{C}(\overline{X})_d$

$\dim I_C(\overline{X})_d = \dim R_d - \dim K[\lambda,\mu]_{3d} =$

$$= \boxed{\binom{d+3}{3} - (3d+1)}$$

we want to compute $\dim \langle F_0, F_1, F_2 \rangle_d$

and check it is equal to $\binom{d+3}{3} - (3d+1)$

$\forall d$

Proof of the Claim: $(G_0, G_1, G_2) \in \operatorname{Ker} \varphi_d \Longrightarrow$

$* \; G_0F_0 + G_1F_1 + G_2F_2 = 0 \Longleftrightarrow$

$$\det \begin{pmatrix} G_0 & G_1 & G_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} = 0$$: the first row is a linear comb of $2^\circ, 3^\circ$ row in $\underline{K(x_0, x_1, x_2, x_3) = Q(R)}$

$G_0 = \frac{a_1}{a_0} x_0 + \frac{b_1}{b_0} x_1 \longrightarrow \frac{a_1x_0 + b_1x_0}{b_0}$ $\qquad b_0 \mid$ numerators

$G_1 = \;'' \; -x_1 \;\; '' \; -x_2 \longrightarrow -\frac{a_1x_1 + b_1x_2}{b_0}$

$G_2 = \;'' \;\; x_2 \;\; '' \;\; x_3 \longrightarrow \frac{a_1x_2 + b_1x_3}{b_0}$

$\Longrightarrow b_0 \mid -b_1F_2, \; b_1F_1, \; -b_1F_0$

F_0, F_1, F_2 are coprime and irreducible

$\Longrightarrow b_0 \mid b_1, \; b_0, b_1$ coprime $\Longrightarrow b_0$ unit $= \pm 1$

and $a_0 = \pm 1 \Longrightarrow$ the coeff. are in R.

$I_{\Gamma}(\bar{X}) = \langle F_0, F_1, F_2 \rangle$

Theorem of Hilbert-Burch.

$$\varphi_d : R_{d-2} \oplus R_{d-2} \oplus R_{d-2} \longrightarrow I_{b}(X)_d$$

$$(G_0, G_1, G_2) \longrightarrow G_0F_0 + G_1F_1 + G_2F_2 \in \langle F_0, F_1, F_2 \rangle_d$$

homog. of deg $\underline{d-2}$

Claim : φ_d is surjective $\forall d$.

$$\operatorname{Ker} \varphi_d = \{ (G_0, G_1, G_2) \mid F_0G_0 + F_1G_1 + F_2G_2 = 0 \}$$

relations of deg d among F_0, F_1, F_2 or

SYZYGYES

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0$$

Laplace : $x_0F_0 - x_1F_1 + x_2F_2 = 0$ 2 syzygies of deg 3 among F_0, F_1, F_2

$x_1F_0 - x_2F_1 + x_3F_2 = 0$

The syzygyes $H_1 = (x_0, -x_1, x_2)$, $H_2 = (x_1, -x_2, x_3)$ generate the syzygies in all degrees.

$$\psi_d : R_{d-3} \oplus R_{d-3} \longrightarrow \operatorname{Ker} \varphi_d$$

$$(A, B) \longrightarrow \boxed{H_1A + H_2B} =$$

$$= (x_0, -x_1, x_2)A + (x_1, -x_2, x_3)B =$$

$$= (x_0A + x_1B, -x_1A - x_2B, x_2A + x_3B)$$

syzygy of deg $d-2$

$$(x_0A + x_1B)F_0 + (-x_1A - x_2B)F_1 + (x_2A + x_3B)F_2 =$$

$$= A(x_0F_0 - x_1F_1 + x_2F_2) + B(x_1F_0 - \ldots) = 0$$

Claim $\boxed{\psi_d \text{ is an isomorphism.}}$

If ψ_d is isom. $\Rightarrow \dim \operatorname{Ker} \varphi_d = 2 \dim R_{d-3} = 2\binom{d-3+3}{3} =$

$$= 2\binom{d}{3}$$

$$\Rightarrow \dim \operatorname{Im} \varphi_d = \dim (R_{d-2} \oplus R_{d-2} \oplus R_{d-2}) - 2\binom{d}{3} =$$

$$= 3\binom{d+1}{3} - 2\binom{d}{3} = \quad \text{to be checked}$$

$$= \binom{d+3}{3} - (3d+1)$$