1.3 Dipole Radiation

The electric and magnetic fields for an oscillating dipole are[3]

$$\begin{aligned}
\mathbf{E}^{(+)}(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\left[3(\hat{\varepsilon}\cdot\hat{r})\hat{r}-\hat{\varepsilon}\right]\left[\frac{d^{(+)}(t_r)}{r^3}+\frac{\dot{d}^{(+)}(t_r)}{cr^2}\right]+\frac{1}{4\pi\epsilon_0}\left[(\hat{\varepsilon}\cdot\hat{r})\hat{r}-\hat{\varepsilon}\right]\frac{\ddot{d}^{(+)}(t_r)}{c^2 r}\\
\mathbf{H}^{(+)}(\mathbf{r},t) &= \frac{c}{4\pi}(\hat{\varepsilon}\times\hat{r})\left[\frac{\dot{d}^{(+)}(t_r)}{cr^2}+\frac{\ddot{d}^{(+)}(t_r)}{c^2 r}\right],
\end{aligned}$$

(dipole radiation fields) (1.42)

[1]Alan Corney, *Atomic and Laser Spectroscopy* (Oxford, 1987).
[2]See Peter W. Milonni and Joseph H. Eberly, *Lasers* (Wiley, 1988), p. 239.
[3]See John David Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), p. 411 or Peter W. Milonni and Joseph H. Eberly, *Lasers* (Wiley, 1988), p. 44.

where $t_r = t - r/c$ is the retarded time, and $\hat{\varepsilon}$ is the polarization unit vector of the applied field (and thus the dipole orientation vector). Only the $1/r$ terms actually transport energy to infinity (i.e., they correspond to radiation), so we can drop the rest to obtain

$$\begin{aligned}
\mathbf{E}^{(+)}(\mathbf{r},t) &\approx \frac{1}{4\pi\epsilon_0 c^2}\left[(\hat{\varepsilon}\cdot\hat{r})\hat{r}-\hat{\varepsilon}\right]\frac{\ddot{d}^{(+)}(t_r)}{r}\\
\mathbf{H}^{(+)}(\mathbf{r},t) &\approx \frac{1}{4\pi c}(\hat{\varepsilon}\times\hat{r})\frac{\ddot{d}^{(+)}(t_r)}{r}.
\end{aligned} \tag{1.43}$$

The energy transport is governed by the Poynting vector, which we can write as

$$\begin{aligned}
\langle\mathbf{S}\rangle &= \mathbf{E}^{(+)}\times\mathbf{H}^{(-)} + \text{c.c.}\\
&= \frac{1}{16\pi^2\epsilon_0 c^3}\frac{|\ddot{d}^{(+)}|^2}{r^2}\left[(\hat{\varepsilon}\cdot\hat{r})\hat{r}-\hat{\varepsilon}\right]\times(\hat{\varepsilon}^*\times\hat{r}) + \text{c.c.}\\
&= \frac{\hat{r}}{16\pi^2\epsilon_0 c^3}\frac{|\ddot{d}^{(+)}|^2}{r^2}\left(1-|\hat{r}\cdot\hat{\varepsilon}|^2\right) + \text{c.c.},
\end{aligned} \tag{1.44}$$

where we have used

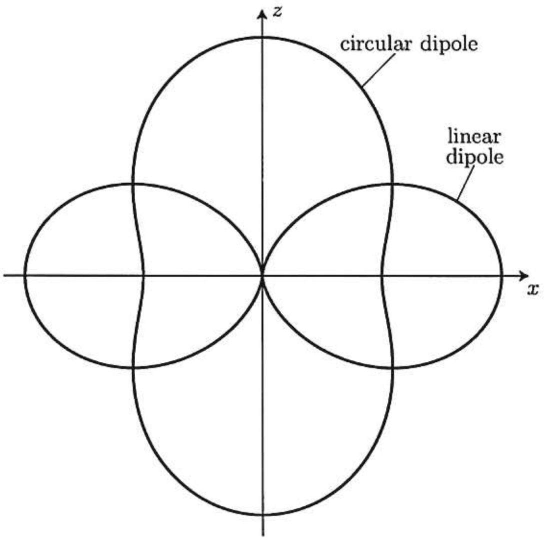
$$\left[(\hat{\varepsilon}\cdot\hat{r})\hat{r}-\hat{\varepsilon}\right]\times(\hat{\varepsilon}^*\times\hat{r}) = \left(1-|\hat{r}\cdot\hat{\varepsilon}|^2\right)\hat{r} \tag{1.45}$$

for the angular dependence.

There are two main possibilities for the polarization vector: the incident light can be linearly or circularly polarized.

1. **Linear polarization** ($\hat{\varepsilon} = \hat{z}$): $1-|\hat{r}\cdot\hat{\varepsilon}|^2 = \sin^2\theta$. This is the usual "doughnut-shaped" radiation pattern for an oscillating dipole.
2. **Circular polarization** ($\hat{\varepsilon} = \hat{\varepsilon}_\pm := \mp(\hat{x}\pm i\hat{y})/\sqrt{2}$): $1-|\hat{r}\cdot\hat{\varepsilon}|^2 = (1+\cos^2\theta)/2$. This is a "peanut-shaped" radiation pattern for a rotating dipole.

Here, θ is the angle from the z-axis, while ϕ is the angle around the azimuth. Note that any arbitrary polarization can be represented as a superposition of these three basis vectors. The (intensity/power) radiation patterns for the linear and circular dipole cases are shown here.



The three-dimensional distributions are generated by sweeping these patterns around the z-axis.

The corresponding electric fields for the dipole radiation are polarized. From Eq. (1.43), we can see that the polarization vector is proportional to $(\hat{\varepsilon}\cdot\hat{r})\hat{r}-\hat{\varepsilon}$. For linear polarization ($\hat{\varepsilon}=\hat{z}$), this factor turns out to be $\sin\theta\,\hat{\theta}$, while for circular polarization ($\hat{\varepsilon}=\hat{\varepsilon}_\pm=\mp(\hat{x}\pm i\hat{y})/\sqrt{2}$), the polarization vector is proportional to $(\cos\theta\,\hat{\theta}\mp i\hat{\phi})e^{\mp i\phi}/\sqrt{2}$.

Now let's define the angular-distribution function via

$$f_{\hat{\varepsilon}}(\theta,\phi) := \frac{3}{8\pi}\left(1-|\hat{r}\cdot\hat{\varepsilon}|^2\right). \tag{1.46}$$

(radiative angular distribution)

For linear and circular polarization, this takes the form

$$\begin{aligned} f_{\hat{z}}(\theta,\phi) &= \frac{3}{8\pi}\sin^2(\theta) \\ f_{\pm}(\theta,\phi) &= \frac{3}{16\pi}\left[1+\cos^2(\theta)\right]. \end{aligned} \tag{1.47}$$

This function has the nice property that it is normalized, and thus represents a probability distribution for photon emission in quantum mechanics:

$$\int f_{\hat{\varepsilon}}(\theta,\phi)\,d\Omega = 1. \tag{1.48}$$

Here, $d\Omega=\sin\theta\,d\theta\,d\phi$ is the usual solid-angle element.

Now we can write the Poynting vector in terms of the angular-distribution function as

$$\langle \mathbf{S}\rangle = \frac{\hat{r}}{3\pi\epsilon_0 c^3}\frac{|\ddot{d}^{(+)}|^2}{r^2} f_{\hat{\varepsilon}}(\theta,\phi). \tag{1.49}$$

The power radiated per unit solid angle is then

$$\frac{dP_{\mathrm{rad}}}{d\Omega} = r^2\langle\mathbf{S}\rangle\cdot\hat{r} = \frac{|\ddot{d}^{(+)}|^2}{3\pi\epsilon_0 c^3} f_{\hat{\varepsilon}}(\theta,\phi), \tag{1.50}$$

and the total radiated power is

$$P_{\mathrm{rad}} = \int d\Omega\,\frac{dP_{\mathrm{rad}}}{d\Omega} = \frac{|\ddot{d}^{(+)}|^2}{3\pi\epsilon_0 c^3} = \frac{e^2|\ddot{x}^{(+)}|^2}{3\pi\epsilon_0 c^3}. \tag{1.51}$$

Of course, the incident intensity is contained implicitly in the electron acceleration $\ddot{x}$.