Trasf. bilineare per analisi di stabilità di sistemi LTI a tempo discreto

$$z = \frac{w+1}{w-1} \qquad w \in \mathbb{C}$$

$z \in \mathbb{C}$

$$z = \frac{1+w}{1-w}$$

Casi di utilizzo

$$p(z) = z^2 - z - 1 \qquad p(z) = 0$$

sistema instabile

$$z_1 = \frac{1-\sqrt{5}}{2} \simeq -0{,}618$$

$$z_2 = \frac{1+\sqrt{5}}{2} \simeq +1{,}618$$

$$z^2 - z - 1 = 0 \longleftarrow z = \frac{w+1}{w-1}$$

$$\left(\frac{w+1}{w-1}\right)^2 - \left(\frac{w+1}{w-1}\right) - 1 = 0 \qquad / (w-1)^2$$

$$w \neq 1$$

$$(w+1)^2 - (w+1)(w-1) - (w-1)^2 = 0$$

$$-w^2 + 4w + 1 = 0 \qquad / (-1)$$

$$w^2 - 4w - 1 = 0$$

"tempo continuo"

criterio di RH

$q(w) \quad w^2 - 4w - 1$

$$
\begin{array}{l r|l l}
1v & 2 & +1 & -1 \\
 & 1 & -4 & \\
1p & 0 & -1 &
\end{array}
$$

$1v \quad 1p$

1 radice $Re(w_1) > 0$

1 radice $Re(w_2) < 0$

$\boxed{\text{Casi particolari}}$

a $\quad \exists \, \bar{z} = -1 \rightarrow p(\bar{z}) = 0$

$$p(z) = (z+1)^2 (z+10)$$

$$z = \frac{w+1}{w-1}$$

$$\left(\frac{w+1}{w-1} + 1\right)^2 \left(\frac{w+1}{w-1} + 10\right) = 0 \quad (w-1)^3$$

$$q(w) = 0 \Longrightarrow 4w^2 (11w - 9) = 0$$

$w = + 9/11 \quad > 0$

$P(z)$ radice doppia in $z = -1$

$q(w)$ ha radice doppia in $w = 0$

radici in $z = -1$ di $P(z)$ $\longleftrightarrow$ radici in $w = 0$ per $q(w)$

stessa molteplicità

$$q(w) = 44w^3 - 36w^2$$

$$\begin{array}{c|cc}
3 & 44 & 0 \\
2 & -36 & 0 \\
1 & & \\
0 & &
\end{array}$$

(b) $p(z)$ ha radici in $z=+1$

$$p(z) = (z-1)^2\left(z+\frac{1}{2}\right) \qquad \Leftarrow \quad z = \left(\frac{w+1}{w-1}\right)$$

grado 3 $\qquad$ radice doppia in $z=+1$

$$\left(\frac{w+1}{w-1}-1\right)^2\left(\frac{w+1}{w-1}+\frac{1}{2}\right)=0 \Big/ (w-1)^3 \qquad w\neq 1$$

$$4(3w+1)=0 \qquad q(w)=0$$

grado 1 $\qquad 3-2=1$

$p(z)$ grado n_z

$p(z)$ ha radice in $z=+1$ $\iff$ $q(w)$ ha grado $n_z - k$

$z=+1$ moltepl. k

$q(w) \to \bar{w} = -1/3 \qquad \mathrm{Re} < 0 \iff z = -\frac{1}{2}$

$|\cdot| < 1$

$\boxed{\text{Stabilità al variare di un parametro}}$

$$
\begin{cases}
x_1(k+1) = 2x_1(k) - 2x_2(k) + u(k) \\
x_2(k+1) = 2x_1(k) - \gamma x_2(k) \\
x_3(k+1) = -2x_1(k) + \gamma x_2(k) \\
y(k) = x_3(k)
\end{cases}
\qquad \gamma \in \mathbb{R}
$$

Come varia la stabilità al variare di γ?

$$
p(z) = \det(zI - A) = \begin{vmatrix} z-2 & 2 & 0 \\ -2 & z+\gamma & 0 \\ 2 & -\gamma & z \end{vmatrix} =
$$

$$
= z\left[z^2 + (\gamma-2)z + 2(2-\gamma) \right]
$$

$\lambda_1 = 0$ indipendente da γ

dipende da γ

$$z^2+(\gamma-2)z+2(2-\gamma)=0 \Leftarrow z=\left(\frac{w+1}{w-1}\right)$$

$$\left(\frac{w+1}{w-1}\right)^2+(\gamma-2)\left(\frac{w+1}{w-1}\right)+2(2-\gamma)=0 \qquad (w-1)^2$$

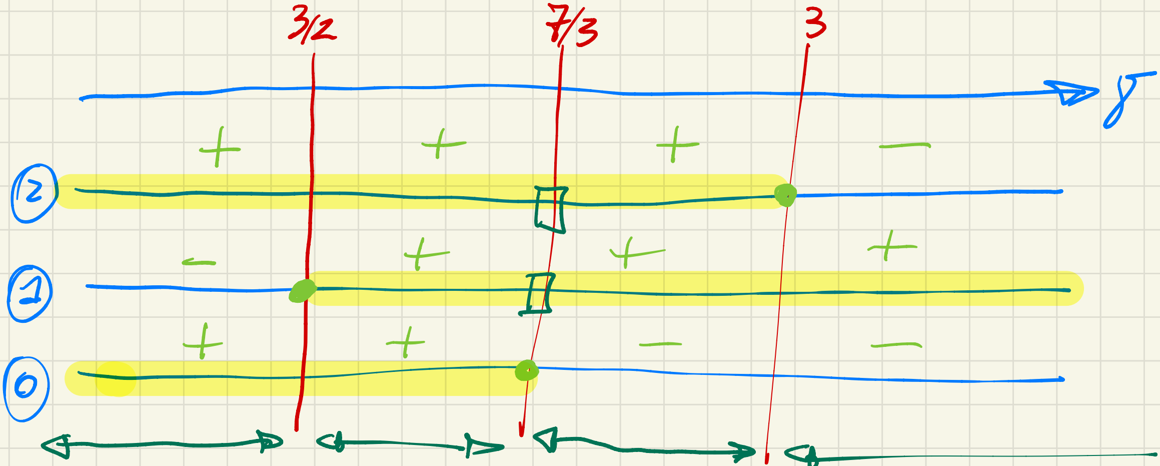
$$(3-\gamma)w^2+2(2\gamma-3)w+(7-3\gamma)=0$$

$$\begin{array}{c|ll} 2 & (3-\gamma) & (7-3\gamma) \\ 1 & 2(2\gamma-3) \leftarrow \gamma=\frac{3}{2} & \\ 0 & (7-3\gamma) & \end{array}$$

$$\text{(2)} \quad 3-\gamma \geq 0 \rightarrow \gamma \leq 3$$

$$\text{(1)} \quad 2\gamma-3 \geq 0 \rightarrow \gamma \geq \frac{3}{2}$$

$$\text{(0)} \quad 7-3\gamma \geq 0 \rightarrow \gamma \leq \frac{7}{3}$$



$\gamma < 3/2$	$3/2 < \gamma < 7/3$	$7/3 < \gamma < 3$	$3 < \gamma$
2v	2p	1p, 1v	2v
sistema INST.	sistema AS. STAB.	sistema INST	sistema INST.

$\gamma = \frac{3}{2}$ 2 radici imm. pure in w

2 radici con $Re(w)=0$ in $q(w)$

$\Downarrow$

2 radici a $|\cdot|=1$ in z, compl. coniug

(sulla circonferenza di raggio 1 e centro 0)

$\downarrow$

stab. semplice

$\gamma = 7/3 \rightarrow w=0 \rightarrow z=-1 \Rightarrow$ sempl. STAB

$\gamma = 3 \longrightarrow q(\omega)$ scende di grado

$z \to 1 \implies z = +1$ con molt. 1

$Re(\,) > 0 \longrightarrow$ INST.