$X \subseteq \mathbb{P}^3$ skew cubic $\quad X = V_p(F_0, F_1, F_2) \quad R = K[x_0, x_1, x_2, x_3]$

$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \qquad I_a(X) = \langle F_0, F_1, F_2 \rangle$

$\forall d \geq 0 \qquad I_a(X)_d$ is gen. by F_0, F_1, F_2 :

$\varphi_d : R_{d-2} \oplus R_{d-2} \oplus R_{d-2} \twoheadrightarrow I_a(X)$ surjective

$\psi_d : R_{d-3} \oplus R_{d-3} \xrightarrow{\sim} \ker \varphi_d$ isomorphism.

$0 \to R \oplus R \xrightarrow{\psi} R \oplus R \oplus R \xrightarrow{\varphi} \boxed{I_a(X)} \to 0$ exact

$e_0', e_1' \longrightarrow H_1, H_2$

free module of rk 2 $\qquad$ free module of rk 3 over R

$e_0, e_1, e_2 \longrightarrow F_0, F_1, F_2$

free resolution of $I_a(X)$: exact sequence of R-modules, which are free

It is a <u>minimal</u> free resolution : the matrices representing ψ, φ w.r. to the canonical bases don't contain constants

$\psi \rightsquigarrow$ matrix M

$\varphi \rightsquigarrow (F_0, F_1, F_2)$

Irreducible topological spaces

X topological space

X is <u>irreducible</u> if $X \neq \emptyset$ and

if $X = X_1 \cup X_2$, X_1, X_2 closed in X $\Longrightarrow$ either $X = X_1$ or $X = X_2$

X cannot be written as union of 2 proper closed subsets.

X connected if $X = X_1 \cup X_2$, X_1, X_2 closed <u>and</u> $X_1 \cap X_2 = \emptyset$ $\Longrightarrow$ $X = X_i$ for some $i = 1, 2$

If X is irred. $\Longrightarrow$ X is connected

$\not\Longleftarrow$

Equivalent def. using open subsets:

X is irreducible $\Longleftrightarrow$ $X \neq \emptyset$ and if U, V are open and $\neq \emptyset$, then $U \cap V \neq \emptyset$

By def. $\emptyset$ is not irreducible.

X reducible $\Longrightarrow$ $X = X_1 \cup X_2$, X_1, X_2 closed $X_1 \subsetneq X, X_2 \subsetneq X$

$\Longrightarrow$ $U \neq \emptyset, V \neq \emptyset$ open $U \cap V = \emptyset$

$\mathbb{R}^n$, euclidean top.: not irreducible

$\mathbb{C}^n$

Prop. X is irreducible $\iff$ every $U \subseteq X$ open, $U \neq \emptyset$, is dense in X: $\overline{U} = X$.

Pf. "$\Rightarrow$" Assume X irreducible, $U \neq \emptyset$ open.

Take $P \in X$: we have to check that $P \in \overline{U}$; take I_P open neighborhood of P, claim $I_P \cap U \neq \emptyset$.

X irred; I_P open $\neq \emptyset$: $P \in I_P$
U open $\neq \emptyset$ $\Big\} \Rightarrow U \cap I_P \neq \emptyset$

"$\Leftarrow$" Assume any $U \neq \emptyset$ open is dense.

U, V non-empty open subsets

$U \ni P$: U is an open nbhd of P

$\overline{V} = X$ $\quad P \in \overline{V} \Rightarrow U \cap V \neq \emptyset$.

Examples 1) $X = \{p\} \implies$ X is irreducible

The closed subsets of X are only $\emptyset$, X

2) $\mathbb{A}^1_K$, K infinite field $\implies \mathbb{A}^1_K$ is irreducible

The closed subsets are $\to \emptyset$, $\to \mathbb{A}^1_K$, $\to$ finite sets

$\mathbb{A}^1$ is not union of 2 proper closed sets

If K is finite, $\mathbb{A}^1_K$ is reducible.

3) $f : X \longrightarrow Y$ continuous map of top spaces

If X is irred., f surjective $\implies$ Y is irred.

Pf. $U, V \subseteq Y$ open, $\neq \emptyset$

$f^{-1}(U), f^{-1}(V)$ open, non empty because f is surj.

X irred. $\implies f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$, $p \in f^{-1}(U) \cap f^{-1}(V)$

$f(p) \in U \cap V \implies U \cap V \neq \emptyset$

4) X top. space, $Y \subseteq X$ top. space with the induced topology

<u>Y is irreducible</u>:

$Y = (Z_1 \cap Y) \cup (Z_2 \cap Y) = (Z_1 \cup Z_2) \cap Y \implies Z_1 \cup Z_2 \supseteq Y$

Z_1, Z_2 closed in X

Y is irred. $\implies$ either $Y = Z_1 \cap Y$ or $Y = Z_2 \cap Y$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$Y \subseteq Z_1 \qquad\qquad Y \subseteq Z_2$$

Y is irred $\iff$ if $Y \subseteq Z_1 \cup Z_2$, Z_1, Z_2 closed in X then $Y \subseteq Z_1$ or $Y \subseteq Z_2$.

using open sets: $Y \cap U \neq \emptyset$, $Y \cap V \neq \emptyset$ $\implies (Y \cap U) \cap (Y \cap V) \neq \emptyset$

$$(Y \cap U) \cap (Y \cap V) = Y \cap (\underline{U \cap V})$$

$f: X \to Y$ continuous $\implies$

$\boxed{\text{if } X \text{ is irreducible} \implies f(X) \text{ is irreducible}}$

Prop. X top. space, $Y \subseteq X$, $\overline{Y} \subseteq X$

Y is irreducible $\iff$ $\overline{Y}$ is irreducible

Pf: $U \subseteq X$ open, $U \cap Y \subseteq U \cap \overline{Y}$

we observe that $\boxed{U \cap Y \neq \emptyset \iff U \cap \overline{Y} \neq \emptyset}$

if $U \cap Y = \emptyset$, if $P \in U \cap \overline{Y}$ $\Longrightarrow$ any open nbhd of P intersects Y, U is an open nbhd of P so $U \cap Y \neq \emptyset$.

Using the characterization w. open subsets

$\Longrightarrow$ Y is irred. iff $\overline{Y}$ is irred.

<u>Cor.</u> X irred. top. space, $U \subseteq X$ open, $U \neq \emptyset$

then $\overline{U} = X$ so $\overline{U}$ is irreducible $\Longrightarrow$

U is irreducible.

$\mathbb{A}^n_K$, $X \subseteq \mathbb{A}^n_K$ affine algebraic variety

X is a top. space w. Zariski topology

Prop. X is irreducible $\Longleftrightarrow$ $I(X)$ is a prime ideal

Pf. " $\Rightarrow$ " Ass. X is irreducible; Take $F, G \in K[x_1, \dots, x_n]$ s.t. $FG \in I(X)$, we have to deduce that either $F \in I(X)$ or $G \in I(X)$

$V(F) \cup V(G) = V(FG) \supseteq V(I(X)) = X$

closed in $\mathbb{A}^n_K$ — because X is closed in $\mathbb{A}^n_K$

$\Rightarrow$ either $X \subseteq V(F)$ | or $X \subseteq V(G)$

$\forall P \in X \quad F(P) = 0$ | $\Rightarrow G \in I(X)$

$\Rightarrow F \in I(X)$

therefore $I(X)$ is prime

" $\Leftarrow$ " Assume $I(X)$ is prime

$X = X_1 \cup X_2$, X_1, X_2 closed: we to prove that either $X_1 = X$ or $X_2 = X$. Assume that $\boxed{X_1 \subsetneq X}$;

$I(X) = I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$

$I(X_1) \supsetneq I(X)$ otherwise: if $I(X_1) = I(X) \Rightarrow V(I(X_1)) = X_1$ closed

$\exists\, F \in I(X_1)$, $\boxed{F \notin I(X)}$ $\qquad V(I(X)) = X$

To prove that $X_2 = X$, we will check that $I(X_2) \underset{\subseteq}{=} I(X)$.

If $G \in I(X_2)$ $\qquad FG \in I(X_1) \cap I(X_2) = I(X)$ prime

$F \notin I(X) \Longrightarrow G \in I(X)$

Therefore $I(X_2) \subseteq I(X) \Longrightarrow X_2 = X$

In $\mathbb{P}^n_K$: $X \subseteq \mathbb{P}^n_X$ projective algebraic set

Prop. $I_h(X)$ is prime $\iff$ X is irreducible.

Pf. Similar using this fact:

Lemma $P \subseteq K[x_0 \dots x_n]$ homog.

P is prime $\iff$ if F, G are homog. poly s.t. $FG \in P$ then $F \in P$ or $G \in P$

Pf of Lemma "$\Rightarrow$" clear

"$\Leftarrow$" H, K polyn. s.t. $\boxed{HK \in P}$

$H = H_0 + H_1 + \dots + H_d$ $\quad d = \deg H$ $\quad H_d \neq 0$

$K = K_0 + K_1 + \dots + K_e$ $\quad e = \deg K$ $\quad K_e \neq 0$

$$HK = \underbrace{H_0K_0} + \underbrace{(H_0K_1 + H_1K_1)}_{\deg 1} + (H_0K_2 + H_1K_1 + H_2K_0) + \dots + \underbrace{H_d K_e}_{\deg d+e}$$

$\deg 2$, $\neq 0$

P homog., H_dK_e is a homog. comp. of $HK \in P$

$\Rightarrow H_d K_e \in P$ $\Rightarrow$ $H_d \in P$ or $K_e \in P$

If $H_d \in P$ $\quad \underbrace{HK}_{\in P} - \underbrace{H_d K}_{\in P} \in P$ $\quad \underbrace{(H - H_d)}K \in P$

$H - H_d$ has a lower number of non-zero homog. components: we can use induction on the number of non-zero homog. components of the factors

Examples

1) $\mathbb{A}^n_K$, $\mathbb{P}^n_K$ For $n=1$, $\mathbb{A}^1_K$ irred $\Leftrightarrow K$ is infinite

$\forall n \geq 1$, an. K is infinite: $\mathbb{A}^n_K$, $\mathbb{P}^n_K$ are irred.

Because, if K is infinite: $I(\mathbb{A}^n) = (0)$ and

$(0) \subseteq K[x_1, \dots, x_n]$ is a prime ideal, because $K[x_1 \dots x_n]$ is an integral domain. $\Rightarrow \mathbb{A}^n$ irred.

$\overline{\mathbb{A}^n_K}$ projective closure is $\mathbb{P}^n_K$

$\mathbb{A}^n_K$ irred. $\Rightarrow \overline{\mathbb{A}^n} = \mathbb{P}^n_K$ is irred.

or

$I_h(\mathbb{P}^n) = (0)$

2) $X \subseteq \mathbb{P}^n$ proj. alg. set, $C(X)$ cone of X

$p: \mathbb{A}^{n+1}_K \setminus \{0\} \longrightarrow \mathbb{P}^n_K$ $(a_0, \dots, a_n) \longrightarrow [a_0, \dots, a_n]$

$p^{-1}(X) \cup \{0\}$ $\qquad$ $\cup\!\text{I}$ $\quad X$

$\boxed{I_h(X) = I(C(X))}$ a homog. polyn. F vanishes on $X \Leftrightarrow F$ vanishes on $C(X)$

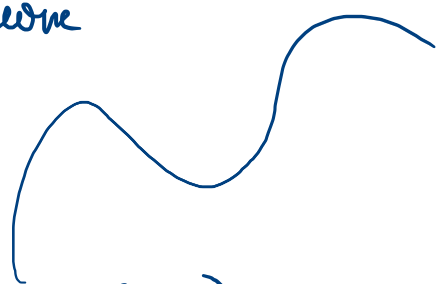
X is irred. $\Leftrightarrow C(X)$ is irred.

3. $X \subseteq \mathbb{A}^n$ hypersurface, F reduced equation

$I(X) = (F)$; if F is irreducible $\Rightarrow (F)$ is a prime ideal $\Rightarrow X$ is irreducible

$n=2$ hypersurface = curve

$F \in K[x,y]$ $\quad X = V(F)$ is not union of 2 proper alg. sets



The same in $\mathbb{P}^n$: $\quad X = V_p(F)$, F irred.

$\Longrightarrow X$ is irred.

4) Assume K algebraically closed:

any prime ideal is radical

$$\begin{array}{ccc}
\{\text{affine alg. sets in } \mathbb{A}^n_K\} & \longleftrightarrow & \{\text{radical ideals in } K[x_1 \dots x_n]\} \\
\cup\!\mid & & \cup\!\mid \\
\{\text{irreducible alg. sets in } \mathbb{A}^n_K\} & \longleftrightarrow & \{\text{prime ideal}\}
\end{array}$$

$$\{\text{irreducible projective alg. subsets in } \mathbb{P}^n_K\} \longleftrightarrow \{\text{homog. non-irrelevant prime ideals}\}$$

|| Any algebraic set is a finite union of irreducible alg. sets

We will prove such kind of property in noetherian topological spaces.

Def. X top. space. X is noetherian if it satisfies the equiv. conditions:

(i) ascending chain condition for open subsets:

$U_0 \subseteq U_1 \subseteq \dots \subseteq U_r \subseteq \dots$

open subsets $\Rightarrow$ it is stationary: $\exists\, m$ s.t.

$U_m = U_{m+i}$, $\forall\, i \geq 0$

(ii) descending chain cond. for closed subsets:

$X_0 \supseteq X_1 \supseteq \dots \supseteq X_r \supseteq \dots$

closed is stationary

(iii) any set $\neq \emptyset$ of open subsets of X has a maximal element

(iv) any $\neq \emptyset$ set of closed subsets in X has minimal element

$\overline{\mathbb{A}^n}$ $\quad \mathbb{A}^n = U_0 \subseteq \mathbb{P}^n$

$Y \subset X$ $\quad Y$ irred. $\iff \overline{Y}$ irred.

$\mathbb{A}^n = U_0 \subseteq \mathbb{P}^n$ $\quad U_0$ irred. $\iff \overline{U_0}$ irred.

$\mathbb{A}^n \longrightarrow U_0 \subset \mathbb{P}^n$ $\quad U_0 = \mathbb{A}^n$, $\overline{U_0} = \mathbb{P}^n$

homeom.