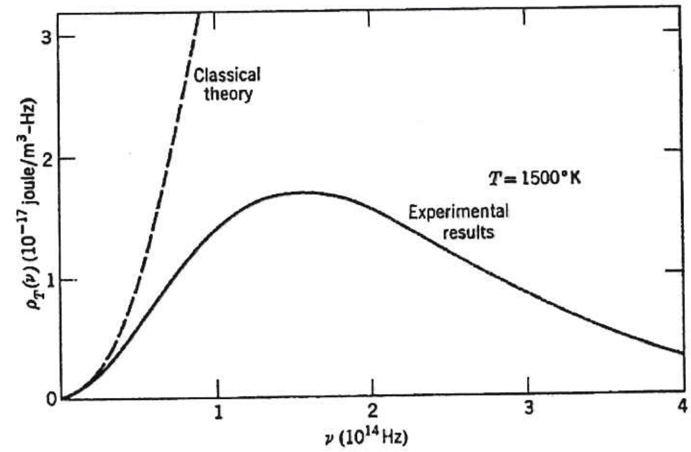
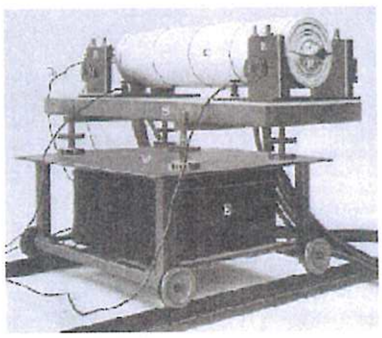
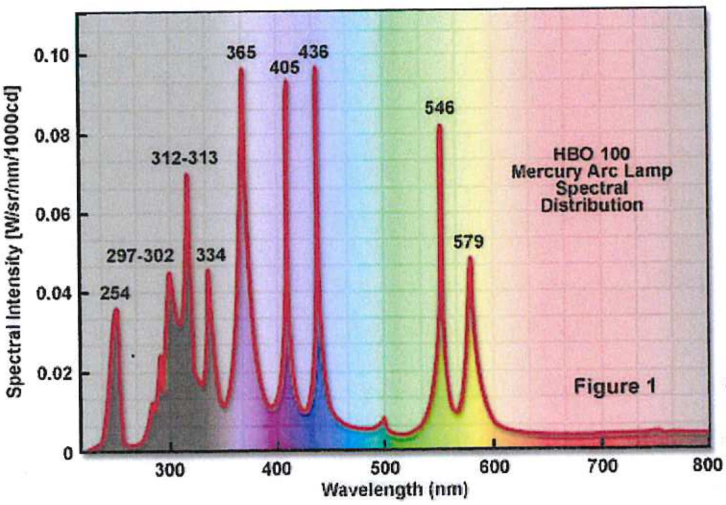
LECTURE #2A

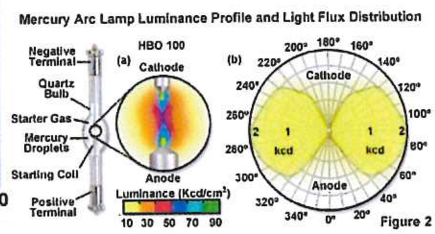
FROM THE BLACKBODY RADIATION TO THE FERMI GOLDEN RULE

AS USUAL WE START FROM EXPERIMENTAL FACTS THAT CANNOT BE EXPLAINED BY THE CLASSICAL E.D.: THE BLACKBODY RADIATION AND THE INTENSITY OF THE SPECTRAL EMISSION (ABSORPTION) LINES.









LET START WITH THE CLASSICAL E.D. CALCULATIONS OF THE BLACKBODY RADIATION. THESE CALCULATIONS LEADS TO THE TO THE RAYLEIGHT-JEANS FORMULA WHERE THE SPECTRAL RADIANCY IS GIVEN BY

$$du = u_T(\nu)d\nu = \frac{8\pi\nu^2 K_B T}{C^3} d\nu$$

AT HIGH FREQUENCY $u_T(\nu)$ DIVERGES GIVING RISE TO THE ULTRAVIOLET CATASTROPE. IN THE FOLLOWING WE FIRST START BY DERIVING THE CLASSICAL LAWS OF THE BLACKBODY (BB) RADIATION AND THEN THE PLANCK EQUATION UNDER THE ASSUMPTION THAT THE ENERGY EXCHANGED BETWEEN THE MATTER AND THE E.M. FIELDS IS QUANTIZED. THIS MODEL ASSUME THAT THE INTERACTION CAN BE REPPRESENTED BY A STATISTIC SET OF LORENTZ OSCILLATORS WITH A SPREA OF FREQUENCIES THAT CAN BE DISCRETE OR CONTINUUM. IS THE ENERGY OF THIS OSCILLATORS THAT IS QUANTIZED AND NOT THE E.M. FIELDS. OF COURSE WE CAN OBTAIN THE SAME RESULT BY QUANTIZING THE E.M. FIELD, I.e. INTRODUCING THE PHOTON (AND WE WILL DO ALSO SO) BUT THIS LEADS TO A CONCEPTUL AND HISTORICAL WRONG CONCLUSION THAT THE BB RADIA-

TAION AND THE PHOTO-ELECTRIC EFFECT CAN BE EXPLAIN ONLY BY QUATIZING THE EM. FIELD AND THEREFORE THEY PROVE THE EXISTENCE AND NATUR OF THE PHOTON. BUT LET'S PROCEED PER ORDER. THE BB RADIATION IS CHARACTERIZED BY EMPIRICAL LAWS:

THE STEFAN-BOLTZMANN WHERE $M_T = \int_0^\infty M_T(\nu) d\nu$ INCREASES RAPIDLY WITH THE TEMPERATUR T (REMEMBER THAT EVERY TIME WE CAN SET A TEMPERATURE FOR A PHYSICAL SYSTEM $\Rightarrow$ THE SYSTEM IS AT EQUILIBRIUM) SO THAT $\boxed{M(T) = \sigma T^4}$

WHERE σ IS A UNIVERSAL CONSTANT HAVIN A VALUE OF $\sim 5.670 \times 10^{8}$ W/m^2K^4

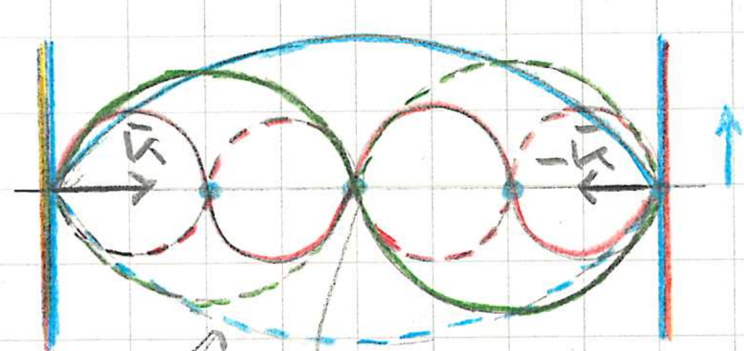
AND THE WIEN DISPLACEMENT LAW THAT STATES THAT THE FREQUENCY CORRESPONDING TO THE MAXIMUM OF THE SPECTRAL RADIANCY IS PROPORTIONAL TO ν $\Rightarrow \nu \propto T$ OR $w = \lambda_{MAX} T$ WHERE w IS THE WIEN CONSTANT.

THE INTENSITY AND SPECTRUM OF BB RADIATION DEPENDS ONLY ON THE TEMPERATURE, i.e. BB RADIATION EXHIBITS A DISTRIBUTION IN ENERGY CHARACTERISTIC OF T AND IT DOES NOT DEPEND UPON THE PROPERTIES OF THE CAVITY. YOU MAY BE FAMILIAR WITH THE STANDARD DERIVATION OF THE

RAYLEIGH-JEANS AND PLANCK RADIATION LAWS BUT IT IS WORTH REPEATING BECAUSE BB RADIATION IS FUDAMENTAL FOR THE QUATUM MECHANICS AND QUATUM PHYSICS. CONSIDER A LARGE CUBICAL CAVITY (ANY OTHER GEOMETRY WILL WORK FINE) OF VOLUME a^3 (WITH a = EDGE OF THE CUBE AND $a \gg \lambda$ BEING λ THE LONGEST WAVELENGTH OF INTEREST). THE CAVITY IS FILLED WITH E.M. RADIATION IN THERMODYNAMIC EQUILIBRIUM. THE PURPOSE OF THE CAVITY IS TO CONFINE THE RADIATION LONG ENOUGH TO REACH THE EQUILIBRIUM. THE WHOLE OF THE CAVITY MUST HAVE NON-ZERO CONDUCTIVITY OTHERWISE THEY WILL BE TRASPARENT.

- OBSERVATION FOR WALLS HAVING NON-ZERO CONDUCTIVITY, THE EQUILIBRIUM ELECTRIC FIELD STRENGTH AT THE WALL IS $\bar{E} = 0$ BECAUSE "MODES" WITH $\bar{E} \neq 0$ AT THE WALL ARE LOSSY. ONLY THOSE STANDING WAVES WITH $\bar{E} = 0$ AT THE WALL WILL PERSIST AFTER SOME TIME $t \gg a/c$ (C: SPEED OF LIGHT).

LET START WITH THE DEFINITION OF STANDING WAVE-MODES. FOR EXAMPLE, CONSIDER ALL STANDING WAVES WHOSE WAVE NORMALS POINT IN THE $\hat{x}$ DIRECTION.



WE CAN ENUMERATE ALL POSSIBLE STANDING-WAVE MODES IN THE CAVITY. FOR EXAMPLE CONSIDER ALL STANDING-WAVE MODES WITH $\bar{k} \parallel \hat{x}$ AND $\perp$ TO THE CAVITY WALL. WE NOTICE THAT THE MODE IN THE EXAMPLE IS LINEALY POLARIZED AND IS MADE OF PROGRESSIVE AND REGRESSIVE WAVE. THE STANDING WAVES CORRESPOND TO $\boxed{n\lambda_x/2 = a}$

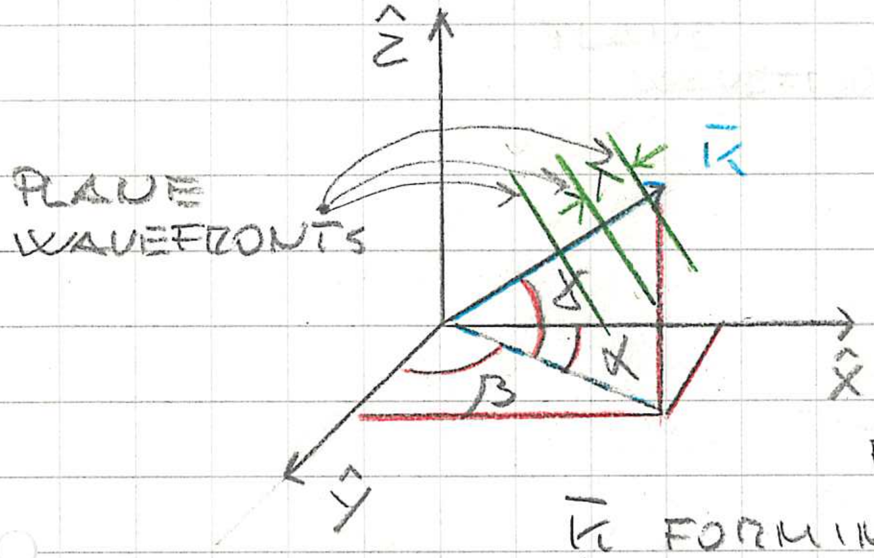
THE BOUDARY CONDITIONS $\bar{E}=0$ AT $x=0$ AND $x=a \Rightarrow$ ONLY WAVES HAVING THE DISCRETE λ CORRESPONDING TO $\frac{\lambda_x}{2}=a$; $\frac{2\lambda_x}{2}=a$, $\frac{3\lambda_x}{2}=a$, ... CAN HAVE NON-ZERO AMPLITUDE $\Rightarrow \frac{n\lambda_x}{2}=a$. IN 3D

$$\frac{n_x\lambda_x}{2}=a;\quad \frac{n_y\lambda_y}{2}=a;\quad \frac{n_z\lambda_z}{2}=a$$

HAVE NON-ZERO AMPLITUDE.

WHAT ABOUT WAVES WHOSE $\bar{k}$ IS IN A ARBITRARY DIRECTIONS?

LET BE α, β, γ THE ANGLES BETWEEN $\bar{k}$ AND THE $\hat{x}, \hat{y}, \hat{z}$ CARTESIAN COORDINATES.



THIS FIGURE SHOWS THE $\bar{k}$ VECTOR OF STANDING WAVES PROPAGATING IN A CAVITY (CUBIC FOR EXMPLE), WITH $\bar{k}$ FORMING AN ANGLE α, β, γ WITH $\hat{x}, \hat{y}, \hat{z}$ RESPECTIVEY. FROM THIS FIGURE IT IS CLEAR THAT $\lambda = \lambda_x \cos\alpha$ (λ = WAVE-LENGTH), $\lambda = \lambda_y \cos\beta$, $\lambda = \lambda_z \cos\gamma$. IT IS CLEAR THAT $\lambda_x, \lambda_y, \lambda_z \geq \lambda$. THESE ARE ALSO THE SPACING BETWEEN THE WAVE NODES MEASURED LONG $\hat{x}, \hat{y}, \hat{z}$ RESPECTIVEY. THEREFORE, THE BOUDARY CONDITIONS ARE

$$n_x = \frac{2a}{\lambda_x}, \quad n_y = \frac{2a}{\lambda_y}, \quad n_z = \frac{2a}{\lambda_z} \Rightarrow$$

$$n_x = \frac{2a\cos\alpha}{\lambda}, \quad n_y = \frac{2a\cos\beta}{\lambda}, \quad n_z = \frac{2a\cos\gamma}{\lambda}$$

$$\Rightarrow n_x^2 + n_y^2 + n_z^2 = \left(\frac{2a}{\lambda}\right)^2 \left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right)$$

BEING THE LAST TERM $= 1 \Rightarrow$

$$n_x^2 + n_y^2 + n_z^2 = \left(\frac{2a}{\lambda}\right)^2 \quad \Rightarrow$$

THE PERMITTED FREQUENCIES OF THESE

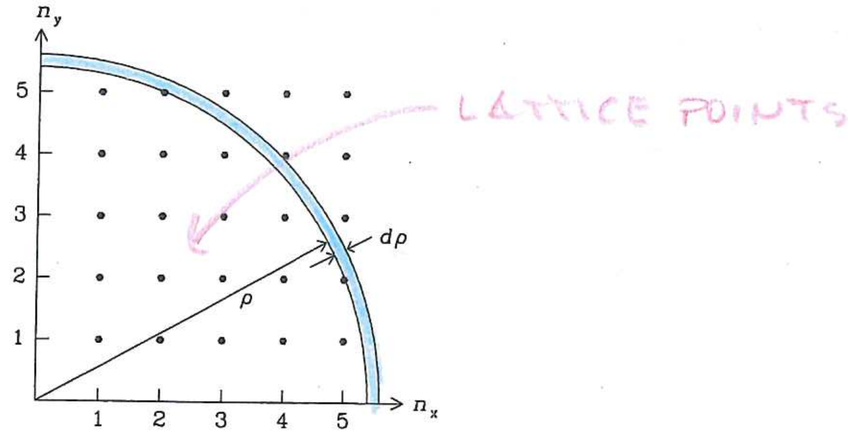
STANDING WAVES IN THE EMPTY SPACE ARE

$$\nu = c/\lambda \Rightarrow \frac{c}{2a}\sqrt{n_x^2 + n_y^2 + n_z^2}$$ FOR ALL

n_x, n_y AND n_z POSITIVE AND INTEGERS.
THE PERMITTED STANDING WAVES WITH A GIVEN λ AND IDENTIFIED BY n_x, n_y, n_z ARE CALLED MODES OF THE CAVITY. THEY CAN BE REPPRESENTED BY THE POINTS OF A 3D CUBIC LATTICE. FOR A VERY LARGE CAVITY THE CUBIC LATTICE CAN BE APPROXIMATED TO A SPHERE (THE NUMBER OF MODES IN THE CUBE OF EDGE a IS APPROXIMATED TO THE NUMBER OF MODES IN A SPHERE OF RADIUS a FOR a VERY LARGE COMPARED TO λ) OR MORE RIGOROUSLY THE PERMITTED STANDING-WAVE MODES CAN BE REPPRESENTED AS A LATTICE IN THE POSITIVE OCTANT OF A SPACE WHOSE AXIS ARE n_x, n_y, n_z. EACH POINT OF THE LATTICE REPPRESENTS ONE POSSIBLE MODE OF THE EM RADIATION. THE SPACE DENSITY OF POINTS IN THIS LATTICE IS UNITY, SO THE NUMBER OF POINTS IN ANY VOLUME IS EQUAL TO THAT VOLUME, THUS LET ρ BE THE RADIAL COORDINATE IN (n_x, n_y, n_z)-SPACE $\Rightarrow$

$$\rho^2 = n_x^2 + n_y^2 + n_z^2$$

AND $\nu = \dfrac{c\rho}{2a}$



The (x,y) plane in the imaginary space whose axes are (n_x, n_y, n_z). Permitted standing waves in the (n_x, n_y) plane are indicated by dots at positive integer values of these axes.

THE NUMBER $N(\nu)d\nu$ OF MODES HAVING A FREQUENCY IN THE RANGE $\nu + d\nu$ IS THE VOLUME OF THE SFERICAL OCTANT SHELL BETWEEN ρ AND $\rho + d\rho$ MULTIPLIED BY 2 (TWO POLARIZATION)

$$N(\nu)d\nu = \frac{4\pi\rho^2 d\rho}{8} \times 2$$

POLAR.

SPHERICAL SHELL VOLUME

OCTANT

$$\Rightarrow N(\nu)d\nu = \pi\left(\frac{2a\nu}{c}\right)^2 \frac{2a}{c}\,d\nu$$

AT T (THERMODYNAMIC EQUILIBRIUM) EACH MODE HAS AVERAGE ENERGY

$\langle U_M \rangle = k_B T$, ACCORDING TO THE

CLASSICAL BOLTZMANN LAW (BUT NOT ACCORDING TO Q.M.)

IF WE PROCEED USING $K_B T$ FOR U THE SPECTRAL ENERGY DENSITY $u_\nu(T)$ OF THE E.M. RADIATION IN THE FREQUENCY RANGE $\nu + d\nu$ IS THE TOTAL ENERGY OF ALL MODES IN THAT FREQUENCY RANGE DIVIDED BY THE VOLUME a^3 OF THE CAVITY

$$u_\nu(T) = \frac{N(\nu)\,d\nu}{a^3} K_B T \Rightarrow$$

$$u_\nu(T) = \frac{8\pi a^3}{a^3} \frac{\nu^2}{c^3} K_B T = 8\pi K_B T \frac{\nu^2}{c^3}$$

THE SPECTRAL ENERGY DENSITY MUST BE EQUAL TO THE TOTAL FLOW OF SPECTRAL POWER PER UNIT AREA DIVIDED BY THE FLOW SPEED c

$$u_\nu = \frac{1}{c} \int_{4\pi} I_\nu \, d\Omega$$

BEING $d\Omega$ THE DIFFERENTIAL SOLID ANGLE. CALLING THE SPECIFIC IRRADIANCE (INTENSITY) OF THE BLACKBODY RADIATION B_ν AND MAKING USE OF THE FACT THAT THE BB RADIATION IS ISOTROPIC WE GET

$$u_\nu = \frac{1}{c} \int_{4\pi} B_\nu \, d\Omega = \frac{B_\nu}{c} \int_{4\pi} d\Omega = \frac{4\pi}{c} B_\nu \Rightarrow$$

$$\frac{4\pi B_\nu}{c} = \frac{4\pi K_B T \nu^2}{c^3} \Rightarrow \boxed{B_\nu = \frac{2 K_B T \nu^2}{c^2}}$$

WHICH IS THE RAYLEIGH-JEANS LAW FOR THE BB RADIATION SPECTRAL BRIGHTNESS

- OBSERVATION
  - WHEN COMPARED WITH THE BB EMISSION EXPERIMENTAL DATA IT AGREES WITH THE EXPERIMENTS ONLY IN THE LOW FREQUNCY LIMIT.
  - THE BRIGHTNESS IS PROPORTIONAL TO ν^2 BECAUSE THE VOLUME OF THE SPHERICAL SHELL IN 3D IS $\propto$ TO ν^2
  - THE ASSUMPTION THAT ALL MODES HAVE $\langle U \rangle = K_B T$ (THE CLASSICAL ASSUMPTION) BRAKES DOWN AT HIGH FREQUENCY
  - B_ν DIVERGES AT HIGH ν. THIS IS KNOWN AS UV-CATASTROPHE.
  - B_ν IS ISOTIROPIC

- THE PLANCK RADIATION LAW

WE WILL TREAT THIS QUESTION FIRST ASSUMING THE THE INTERACTION BETWEEN THE E.M. RADIATION AND THE MATTER (REPPRESENTED BY LORENTZ OSCILLATORS) IS QUATIZED BECAUSE THE ENERGY OF THE OSCILLATORS IS QUANTIZED AND NOT THE ENERGY OF THE E.M. FIELD. THIS MEANS THAT WE DO NOT ASSUME THE EXISTENCE OF THE PHOTON, ACCORDINGY TO PLANK.

THAN, AS FOR THE MODERN THEORY WE

ASSUME THE PHOTON AND WE WILL DERIVE THE PLANCK'S LAW AGAIN.

- PLANCK'S DERIVATION

IN THE CLASSICAL THEORY, BB RADIATION IS MODELLED AS THE RADIATION EMITTED FROM OSCILLATING CHARGED PARTICLES AT THE MATTER SURFACE. THESE OSCILLATIONS ARE PRODUCED BY THE MOTION RELATED TO THE INTERNAL ENERGY OF THE MATTER, HENCE AT EQUILIBRIUM TO ITS TEMPERUTE.

WE TREAT EACH CHARGE AS A SIMPLE HARMONIC OSCILLATOR

$$\bar{F} = -k(x - x_0)$$

IN 1D. $\bar{F}$ IS DERIVED FROM THE SPRING POTENTIAL ENERGY $V(x) = \frac{1}{2}k(x-x_0)^2 \Rightarrow$

$$x(t) = x_0 + A\cos(\omega t + \phi) \quad \text{AND} \quad \omega = \sqrt{\frac{k}{m}}$$

IN THE HAMILTONIAN FORMALISM THE VALUE OF $\nu = \omega/2\pi$ IS DETERMINED BY THE INITIAL POSITION $x(0)$ AND INITIAL MOMENTUM $\bar{p}(0)$ OF THE OSCILLATOR. THE TOTAL ENERGY

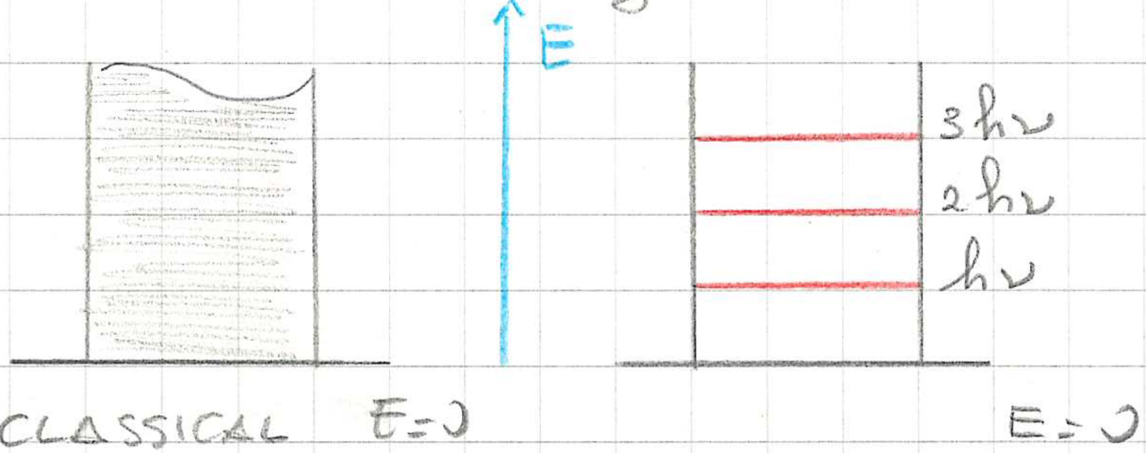
$$E \equiv H = \frac{p^2}{2m} + \frac{1}{2}k(x-x_0)^2.$$

AND IT CAN TAKE ON ANY VALUE, AS DETERMINED BY THE INITIAL CONDITIONS, BUT ONCE THIS VALUE IS SET, IT DOES NOT CHANGE OVER TIME IF THE ENERGY IS CONSERVED.

PLANCK PROPOSED THE FOLLOWING HYPTHESIS.
THE ENERGY OF LORENTZ'S OSCILLATOR IS QUANTIZED => IT CAN ASSUME ONLY DISCRETE ENERGY VALUE IN SUCH A WAY THAT

$$E \propto n\nu \quad \text{WITH } n = 0, 1, 2 \ldots$$

THE CONSTANT OF PROPORTIONALITY IS LABELLED h AND IS KNOWN AS THE PLANCK CONSTANT AND HAS THE VALUE OF $\sim 6.62607 \times 10^{-34}\, m^2 kg/s$



SINCE THERE ARE MANY OSCILLATING CHAGED PARTICLE WE NEED TO CONSIDER A VERY LARGE NUMBER OF IDENTICAL OSCILLATING SYSTEMS.
THIS IS A PROBLEM THAT MUST BE TREATED ON THE BASE OF THE BOLTZMANN'S STATISTICS.
=> THE PROBABILITY THAT STATISTICAL MEANINGFUL SET OF IDENTICAL SYSTEMS AT THE SAME T BUT WITH DIFFERENT INITIAL CONDITIONS WOULD HAVE AN ENERGY PROPORTIONAL TO K_B IS $e^{-E/K_BT} = P$
THAT IS KNOWN AS BOLTZMANN PARTITION

FUNCTION. BY RECALLING THE DEFINITION OF PROBABILITY GIVEN N POSSIBLE EVENTS $\mathcal{E}_1, \mathcal{E}_2 \dots$ $\dots \mathcal{E}_N$ THE PROBABILITY THAN AN EVENT $\mathcal{E}_n$ WILL OCCOUR IS DEFINED TO BE

$$\frac{\text{PROBABILITY THAT } \mathcal{E}_n \text{ OCCURS} = P(\dots}{\text{TOTAL NUMBER OF WAY ANY EVENT CAN OCCUR}}$$

PLANCK APPLIED THIS FORMULA TO THE QUATIZED ENERGY VALUE. LET $\mathcal{E}_n$ AN EVENT THAT MEASURES THE ENERGY OF THE SYSTEM AT T THAT YIELDS AN ENERGY $E = nh\nu$ THE PROBABILITY OF SUCH AN EVENT WILL OCCOUR IS

$$P = \frac{e^{-nh\nu/k_BT}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_BT}}$$

OF COURSE THE SUM OVER ALL THE PROBABILITIES MUST BE 1.

THE DENOMINATOR IS OF THE FORM

$$\sum_{n=0}^{\infty} \left(e^{-h\nu/k_BT} \right)^n$$

AND SINCE $0 < \exp(-h\nu/k_BT) < 1$ IT FOLLOWS THAT THE SUM IS NOTING MORE THAN A GEOMETRIC SERIES FOR WHICH THE FOLLOWING SUMMATION FORMULA APPLIES

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

WITH $0 < r < 1$. APPLYING THIS TO THE PROBABILITY FORMULA WE OBTAIN

$$P = e^{-nh\nu/k_BT}\left(1 - e^{-h\nu/k_BT}\right)$$

SIMILARLY THE AVERAGE ENERGY OF SUCH A COLLECTION OF SYSTEMS IS

$$\bar{E} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nh\nu/k_BT}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_BT}}$$

THESE SUMS CAN BE WORKED OUT WITH A LOT OF ALGEBRA NOT REPORTED HERE TO GIVE

$$\bar{E} = \frac{h\nu}{e^{h\nu/k_BT} - 1}$$