The function j_X is different from 0 only on a finite or countable set of points $(x_n)_n$. We define

$$\chi(t) = \sum_{x_n \le t} j(x_n).$$

We define $F_1 = F - \chi$. The function $F_1 : \mathbb{R} \to [0\,1]$ is continuous and increasing. We know that F_1' is a L^1 (Lebesgue integrable) function and, for all $t_1 < t_2$,

$$\int_{t_1}^{t_2} F_1'(s)\,ds \le F_1(t_2) - F_1(t_1).$$

In particular, for all $t_1 < t_2$,

$$F_1(t_1) - \int_0^{t_1} F_1'(s)\,ds \le F_1(t_2) - \int_0^{t_2} F_1'(s)\,ds.$$

We set

$$G(t) = F_1(t) - \int_0^t F_1'(s)\,ds \qquad \text{and} \qquad F_2(t) = \int_0^t F_1'(s)\,ds.$$

In conclusion

$$F(t) = F_2(t) + G(t) + \chi(t),$$

where F_2 is an absolutely continuous increasing function, G is a continuous increasing function such that $G'(x) = 0$ for almost every $x \in \mathbb{R}$ and χ is a jump-function.

F can be thought as the distribution function (in Italian: funzione di ripartizione) of a random variable X. We have

$$P(X \le t) = F(t),$$

where P is the probability measure associate to X. This random variable has an absolutely continuous density (the function F_2') and a discrete density (the function j_X) but has also a "singular part" (linked to the function G) which cannot be described in term of Lebesgue measure nor in term of discrete random variables.

7 Lesson 8 – March 25$^{\text{th}}$, 2021

7.1 The Hardy-Littlewood maximal function

The content of this paragraph can be (partially) found in [8, Ch. 1] and [7, Ch. 8].

Let us denote by $\mathcal{B}$ the σ-algebra of Borel sets of $\mathbb{R}^d$. Let λ be the Lebesgue measure on $\mathbb{R}^d$ and let ν be a complex measure defined on $\mathcal{B}$. For any ball $B(x, r) = \{y \in \mathbb{R}^d \mid |y - x| < r\}$, we set

$$Q_r\nu(x) = \frac{\nu(B(x,r))}{\lambda(B(x,r))}.$$

Lesson 8 (March 25^{th}, 2021)

Th (Radon - Nikodym)

Let μ is σ-finite positive measure } on $(\Omega, \mathcal{F})$
Let ν a signed or complex measure }

Let $\nu \ll \mu$ $\quad$ ($\forall A \in \mathcal{F}$, $\mu(A) = 0 \Rightarrow |\nu(A)| = 0$)

then $\exists f_0$ measurable s.t.

$\forall A \in \mathcal{F}$, $|\nu(A)| < +\infty \Rightarrow f_0 \chi_A \in L^1_\mu(\Omega)$

and $\nu(A) = \int_A f_0 \, d\mu$

(if ν is a complex measure then $\forall A$ $|\nu(A)| < +\infty$ (density)
so that the thesis is simply
$\exists f_0 \in L^1_\mu(\Omega)$ s.t. $\forall A \in \mathcal{F}$, $\nu(A) = \int_A f_0 \, d\mu$)

Rem. the hypothesis "μ is σ-finite" cannot be avoided

ex. $\mathcal{F}$ = Borelian sets on $\mathbb{R}^d$

let μ be the point-counting measure

$$\mu(A) = \begin{cases} \text{number of points in } A, & \text{if } A \text{ is finite} \\ +\infty & \text{if } A \text{ is infinite.} \end{cases}$$

let ν be the usual Lebesgue measure.

so $\nu \ll \mu$ $\quad$ (if $\mu(A) = 0 \Rightarrow A = \emptyset \Rightarrow \lambda(\emptyset) = 0$)

mind that μ is not $\underbrace{\sigma\text{-finite}}$
means that $\Omega (= \mathbb{R}^d)$ is a countable union of set of finite measure

I show that R-N is true.

Suppose it is true $\quad \exists f_0$ s.t. $\forall A \in \mathcal{B}$

if $\lambda(A) < +\infty$ then $\lambda(A) = \int_A f_0 \, d\mu$

then in particular true for $A = \{x_0\}$

$$0 = \lambda(\{x_0\}) = \int_{\{x_0\}} f_0(x) \, d\mu = f_0(x_0) \cdot \underbrace{\mu(\{x_0\})}_{=1} = f_0(x_0)$$

for all $x_0 \Rightarrow f_0 \equiv 0$ impossible

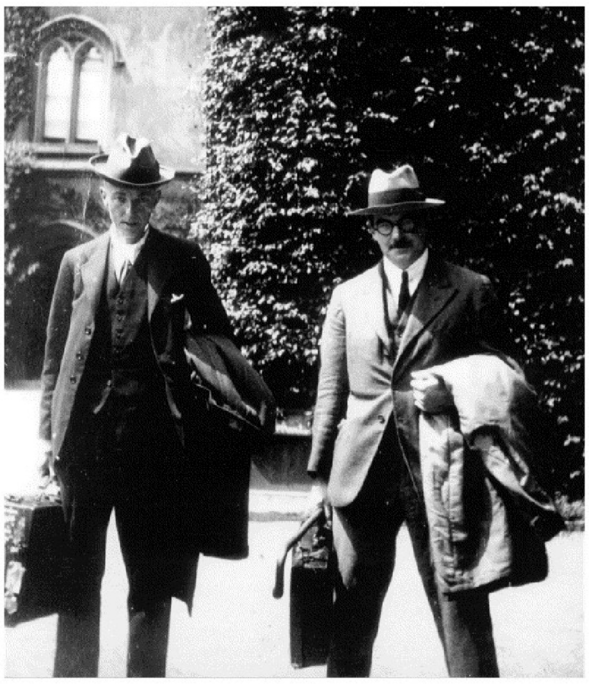


Figure 12: G. H. Hardy and J. E. Littlewood in 1924

Definition 17. *Let $x \in \mathbb{R}^d$. If the limit*

$$\lim_{r \to 0^+} Q_r \nu(x) = \lim_{r \to 0^+} \frac{\nu(B(x,r))}{\lambda(B(x,r))}$$

exists, we call this limit symmetric derivative of ν with respect to λ at the point x and we denote it with $\frac{d\nu}{d\lambda}(x)$.

Remark 14. *Let $f \in L^1_\lambda(\mathbb{R}^n)$ and ν_f such that $\nu_f(A) = \int_A f \, d\lambda$, then*

$$Q_r \nu_f(x) = \frac{\int_{B(x,r)} f \, d\lambda}{\lambda(B(x,r))} \qquad \textit{and} \qquad \frac{d\nu_f}{d\lambda}(x_0) = \lim_{r \to 0^+} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f \, d\lambda.$$

We are interested in conditions guaranteeing the existence of $\frac{d\nu}{d\lambda}$ and also to the value of this quantity.

Definition 18. *Let ν and $|\nu|$ be a complex measure and its total variation, respectively. Let $x \in \mathbb{R}^d$. We define*

$$M_\nu(x) = \sup_{r > 0} \frac{|\nu|(B(x,r))}{\lambda(B(x,r))}.$$

The function $M_\nu : \mathbb{R}^d \to [0, +\infty]$ is called Hardy-Littlewood maximal function of ν.

Theorem 23. *The function M_ν is lower semicontinuous.*

Proof. It is not restrictive to suppose that ν is a positive measure. Proving that M_ν is lower semicontinuous means to show that for all $\alpha \geq 0$, the set

$$E = \{x \in \mathbb{R}^n \mid M_\nu(x) > \alpha\}$$

The Hardy-Littlewood maximal function

def. I denote by $\mathcal{B}$ = Borelian sets = minimal σ-algebra of parts of $\mathbb{R}^d$ containing open sets.

I denote by λ the Lebesgue measure on $\mathcal{B}$

I denote also $B(x,r) = \{ y \in \mathbb{R}^d : |y-x| < r \}$

let ν complex measure

I denote by $Q\nu_r(x) = \dfrac{\nu(B(x,r))}{\lambda(B(x,r))}$

I will call <u>symmetric derivative of ν w.r.t. λ at the point x</u>

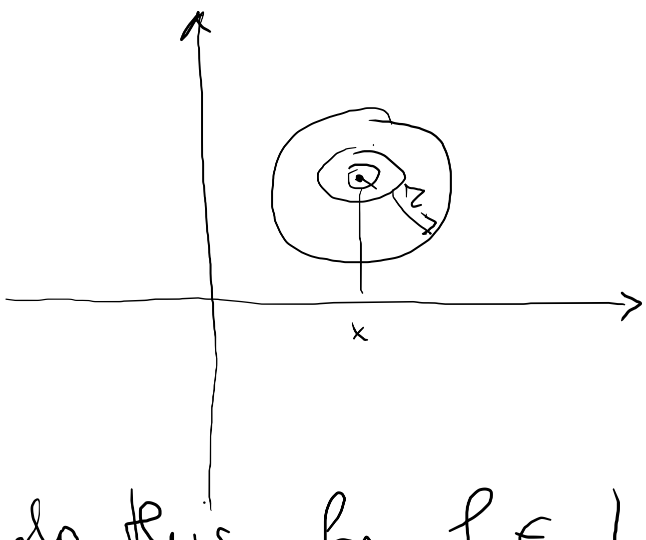
$$\frac{d\nu}{d\lambda}(x) = \lim_{r \to 0^+} Q_{\nu_r}(x) = \lim_{r\to 0^+} \frac{\nu(B(x,r))}{\lambda(B(x,r))}$$

<u>if this limit exists.</u>

Ex. take $f \in L^1(\mathbb{R}^d)$ consider ν_f s.t. $\nu_f(A) = \int_A f \, d\lambda$

symm. derivative

$$\lim_{r \to 0^+} \underbrace{\frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f \, d\lambda}_{\text{integral mean of } f \text{ in the ball } B(x,r)}$$



do this for $f \in L^1(\mathbb{R})$!

$$\lim_{h \to 0^+} \frac{1}{h} \int_{x-h/2}^{x+h/2} f(y)\,dy = f(x) \quad \text{for almost all } x \in \mathbb{R}$$

(theorem on the so called Lebesgue points) def. let ν be a complex measure on $\mathcal{B}$

The function $M_\nu : \mathbb{R}^d \to [0, +\infty]$

$$M_\nu(x) = \sup_{r>0} Q_r|\nu|(x) = \sup_{r>0} \frac{|\nu|(B(x,r))}{\lambda(B(x,r))}$$

($|\nu|$ is the total variation of ν)

$|\nu|$ is positive finite measure on $\mathcal{B}$

M_ν is called the Hardy-Littlewood maximal function

rem if $\nu = \nu_f$ with $f \in L^1(\mathbb{R}^d)$

$$M_{\nu_f}(x) = \sup_{r>0} \frac{\int_{B(x,r)} |f| \, d\lambda}{\lambda(B(x,r))}$$

Theorem M_ν is lower semicontinuous. $\left(M_\nu : \mathbb{R}^d \to [0,+\infty] \right)$

proof. we have to prove that, for all $\alpha \geq 0$ the set $E=\{x\in\mathbb{R}^d : M_\nu(x)>\alpha\}$ is open

Take $x\in E$. then $\displaystyle\sup_{r>0} \frac{|\nu|(B(x,r))}{\lambda(B(x,r))} > \alpha$

then $\exists\, r>0$ s.t. $\displaystyle\frac{|\nu|(B(x,r))}{\lambda(B(x,r))}>\alpha$

so that $\displaystyle\frac{|\nu|(B(x,r))}{\lambda(B(x,r))} > t > \alpha' > \alpha$ $\quad (\exists\, t,\alpha')$

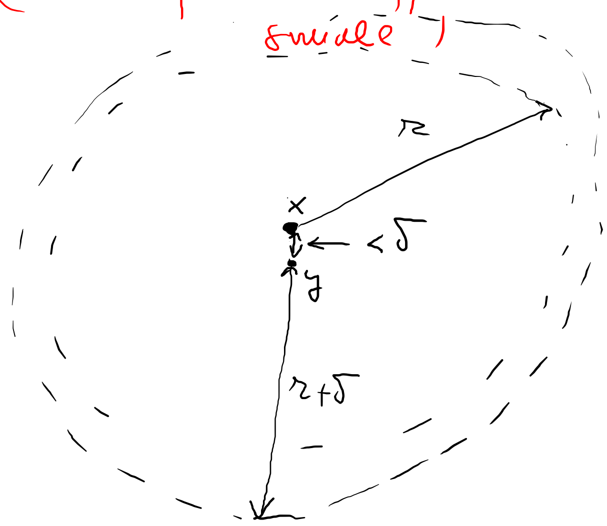
There exists $\delta>0$ s.t. $\displaystyle (r+\delta)^d < \overbrace{\frac{t}{\alpha'}}^{>1} r^d$ $\Rightarrow$ $\displaystyle t > \frac{(r+\delta)^d}{r^d}\cdot\alpha'$

$> r^d$

(true if δ is suff. smalle)

take $y\in\mathbb{R}^d$ s.t. $|y-x|<\delta$

then $B(y,r+\delta) \supseteq B(x,r)$



this is a positive measure

$\Rightarrow |\nu|(B(y,r+\delta)) \geq |\nu|(B(x,r))$

$> t\,\lambda(B(x,r))$

$\displaystyle > \frac{(r+\delta)^d}{r^d}\cdot\alpha'\cdot\lambda(B(x,r))$

$\displaystyle > \underbrace{\alpha'}_{>\alpha}\, \underbrace{\lambda(B(x,r+\delta))}_{\lambda(B(y,r+\delta))}$

λ lebesgue measure

$\displaystyle \frac{\lambda(B(x,r))}{r^d}\cdot(r+\delta)^d = \lambda(B(x,r+\delta))$

conclusion $|\nu|(B(y,r+\delta)) > \alpha\,\lambda(B(y,r+\delta))$

$\Rightarrow y\in E \Rightarrow E$ is open

QED

is an open set. Let $x \in E$. Then $M_\nu(x) > \alpha$ and consequently

$$\sup_{r>0} \frac{\nu(B(x,r))}{\lambda(B(x,r))} > \alpha.$$

Hence there exist $r > 0$ and $t > \alpha' > \alpha$ such that

$$\frac{\nu(B(x,r))}{\lambda(B(x,r))} > t > \alpha' > \alpha.$$

Take now $\delta > 0$ such that

$$(r+\delta)^n < r^n \frac{t}{\alpha'},$$

so that, if $|x-y| < \delta$, then $B(y, r+\delta) \supseteq B(x,r)$ and consequently

$$\begin{aligned} \nu(B(y,r+\delta)) &\geq \nu(B(x,r)) > t\lambda(B(x,r)) \\ &> \alpha' \frac{(r+\delta)^n}{r^n} \lambda(B(x,r)) = \alpha' \lambda(B(x, r+\delta)) = \alpha' \lambda(B(y, r+\delta)). \end{aligned}$$

Finally

$$\frac{\nu(B(y,r+\delta))}{\lambda(B(y,r+\delta))} > \alpha' > \alpha,$$

i. e. we have proved that if $|x-y| < \delta$, then $y \in E$, and consequently E is an open set. □

Corollary 8. *The function M_ν is Lebesgue measurable.*

Lemma 9 (Wiener)**.** *Let W be the union of a finite number of balls $B(x_1, r_1)$, $B(x_2, r_2)$, ..., $B(x_k, r_k)$.*

Then there exists $S \subseteq \{1,\ 2, \ldots,\ k\}$ such that

i) if $i, j \in S$, with $i \neq j$, then $B(x_i, r_i) \cap B(x_j, r_j) = \emptyset$;

ii) $W \subseteq \bigcup_{i \in S} B(x_i, 3r_i)$;

iii) $\lambda(W) \leq 3^n \sum_{i \in S} \lambda(B(x_i, r_i))$.

Proof. The fact that ii) implies iii) is a consequence of the homogeneity property of Lebesgue measure. Let's show i) and ii). It is not restrictive to suppose that

$$r_1 \geq r_2 \geq \ldots \geq r_k.$$

Let $n_1 = 1$. We define

$$A_2 = \{j \in \{n_1+1, \ldots, k\} \bigm| B(x_{n_1}, r_{n_1}) \cap B(x_j, r_j) = \emptyset\}.$$

If $A_2 = \emptyset$, we take $S = \{n_1\}$. If $A_2 \neq \emptyset$, we define $n_2 = \min A_2$. We consider

$$A_3 = \{j \in A_2 \bigm| B(x_{n_2}, r_{n_2}) \cap B(x_j, r_j) = \emptyset\}.$$

If $A_2 = \emptyset$, we take $S = \{n_1,\ n_2\}$. If $A_2 \neq \emptyset$, we define $n_3 = \min A_3$ and we go on with this procedure up to obtaining

$$S = \{n_1,\ n_2, \ldots, n_h\} \quad \text{with} \quad 1 = n_1 < n_2 < \ldots < n_h \leq k.$$

Lemma (Wiener)

Let $B(x_1,r_1) \dots B(x_k,r_k)$ be (open) balls in $\mathbb{R}^d$

then $\exists\ S' \subseteq \{1, 2, \dots k\}$ s.t.

i) $\forall i, j \in S'$ if $i \neq j$ $\quad B(x_i,r_i) \cap B(x_j,r_j) = \emptyset$

ii) $\displaystyle\bigcup_{i=1}^{k} B(x_i,r_i) \subseteq \bigcup_{j \in S'} B(x_j, 3r_j)$

iii) $\displaystyle\lambda\Big(\bigcup_{i=1}^{k} B(x_i,r_i)\Big) \leq 3^d \cdot \underbrace{\sum_{j \in S'} \lambda(B(x_j,r_j))}_{\lambda(\bigcup_{j \in S'} B(x_j,r_j))}$

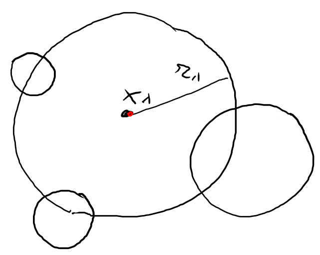
proof. i), ii) $\Rightarrow$ iii) immediate

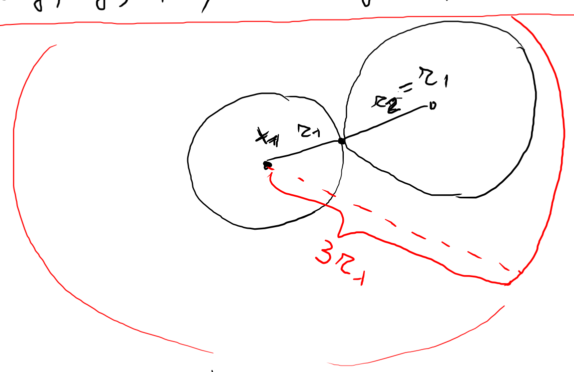
I prove i) and ii)

It is not restrictive to suppose that
$$r_1 \geq r_2 \geq r_3 \geq \dots \geq r_k$$

fix $n_1 = 1$

if $B(x_1,r_1) \cap B(x_j,r_j) \neq \emptyset \quad \forall j = 2, 3, \dots k$





$S' = \{1\}$ $\qquad \displaystyle\bigcup_{i=1}^{k} B(x_i,r_i) \subseteq B(x_1, 3r_1)$

on the contrary

suppose $A_2 = \{ j \in \{2, 3, \dots k\} \mid B(x_1,r_1) \cap B(x_j,r_j) = \emptyset \} \neq \emptyset$

I fix $n_2 = \min A_2$

I consider $A_3 = \{ j \in A_2 \mid B(x_{n_2}, r_2) \cap B(x_j, r_j) = \emptyset \}$

if $A_3 = \emptyset$ then $S' = \{n_1, n_2\}$

otherwise $n_3 = \min A_3$

and so on $\qquad$ I find $1 = n_1 < n_2 < \dots < n_m \leq k$

Now I prove that $\displaystyle B(x_j, r_j) \subseteq \bigcup_{l \in S'} B(x_{n_l}, r_{n_l})$

suppose $n_i < j < n_{i+1}$

$$B(x_{n_i}, r_{n_i}) \cap B(x_j, r_j) \neq \emptyset$$

and $\quad r_{n_i} \geq r_j$

$$\Rightarrow B(x_j,r_j) \subseteq B(x_{n_i}, 3r_{n_i})$$

QED

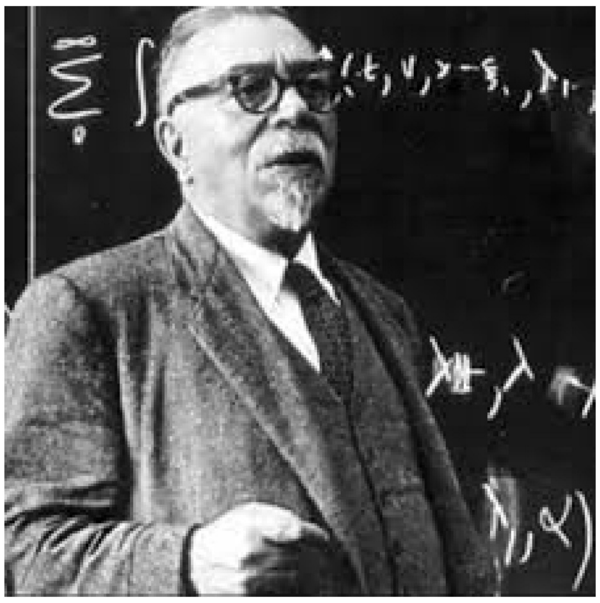


Figure 13: Norbert Wiener (1894–1964)

With such a construction condition i) is verified. Let now $n_i < j < n_{i+1}$. We have $B(x_{n_i}, r_{n_i}) \cap B(x_j, r_j) \neq \emptyset$ and $r_{n_i} \geq r_j$, so that

$$B(x_j, r_j) \subseteq B(x_{n_i}, 3r_{n_i})$$

and condition ii) follows. $\square$

We are now ready to show the main property of Hardy-Littlewood maximal function.

Theorem 24 (Hardy-Littlewood). *Let ν be a complex measure. Let $\alpha > 0$. Then*

$$\lambda(\{x \in \mathbb{R}^d \mid M_\nu(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \cdot |\nu|(\mathbb{R}^d).$$

Proof. Let K be a compact set contained in $E = \{x \in \mathbb{R}^d \mid M_\nu(x) > \alpha\}$ (remember that the set E is measurable). Let $x \in K \subseteq E$. We know that $M_\nu(x) > \alpha$. Then there exists $r_x > 0$ such that

$$\frac{|\nu|(B(x, r_x))}{\lambda(B(x, r_x))} > \alpha.$$

The set $\{B(x, r_x) \mid x \in K\}$ is an open covering of the compact set K. Let

$$B(x_1, r_1),\ B(x_2, r_2), \ldots, B(x_n, r_n),$$

a finite subcovering and let S be the set of indexes given from Wiener's lemma. We have

$$K \subseteq \bigcup_{i=1}^{n} B(x_i, r_i) \subseteq \bigcup_{j \in S} B(x_j, 3r_j).$$

Consequently

$$\begin{aligned} \lambda(K) &\leq \sum_{j \in S} \lambda(B(x_j, 3r_j)) \leq 3^d \cdot \sum_{j \in S} \lambda(B(x_j, r_j)) \\ &\leq 3^d \cdot \frac{1}{\alpha} \cdot \sum_{j \in S} |\nu|(B(x_j, r_j)) \leq 3^d \cdot \frac{1}{\alpha} \cdot |\nu|(\mathbb{R}^d). \end{aligned}$$

Th (Hardy - Littlewood)

Let ν be a complex measure in $\mathbb{R}^d$, let $\alpha > 0$,

then $\lambda(\{x \in \mathbb{R}^d \mid M_\nu(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \cdot |\nu|(\mathbb{R}^d)$

proof. I know that $\{x \in \mathbb{R}^d \mid M_\nu(x) > \alpha\} = E$ is an open set (in particular is a measurable set)

Suppose K is compact in $\mathbb{R}^d$, $K \subseteq E$

let $x \in K$ then $\exists\, r_x > 0$ s.t. $\frac{|\nu|(B(x,r_x))}{\lambda(B(x,r_x))} > \alpha$

$\Downarrow$

$\lambda(B(x,r_x)) < \frac{1}{\alpha}|\nu|(B(x,r_x))$

for all $x \in K$ we have $\underbrace{B(x,r_x)}$

this is an open covering of K, compact

we extract a finite subcovering.

$$K \subseteq \underbrace{B(x_1,r_1) \cup \cdots \cup B(x_n,r_n)}_{\text{we apply Wiener lemma}}$$

$$K \subseteq \bigcup_{j \in S} B(x_j, 3r_j)$$

$$\lambda(K) \leq 3^d \sum_{j \in S} \lambda(B(x_j,r_j))$$

$$\leq 3^d \sum_{j \in S} \frac{1}{\alpha} |\nu|(B(x_j,r_j))$$

$$\leq 3^d \cdot \frac{1}{\alpha} \sum_{j \in S} |\nu|(B(x_j,r_j))$$

but $\leq |\nu|(\mathbb{R}^d)$

p. dij.

$$\lambda(K) \leq 3^d \cdot \frac{1}{\alpha} \cdot |\nu|(\mathbb{R}^d)$$

true for all compact sets inside E $\Rightarrow$ true also for $\lambda(E)$

QED

Remark if $\nu = \nu_f$ for $f \in L^1(\mathbb{R}^d)$

$$\lambda(\{x \in \mathbb{R}^d \mid M_{\nu_f}(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \underbrace{|\nu_f|(\mathbb{R}^d)}_{\int_{\mathbb{R}^d} |f| \, d\lambda}$$

so that

usual statement

$$\lambda(\{x \in \mathbb{R}^d \mid M_{\nu_f}(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \cdot \|f\|_{L^1}$$

It is possible to prove that

if $f \in L^p(\mathbb{R}^d)$ with $1 < p \leq \infty$

then $M_{\nu_f} \in L^p(\mathbb{R}^d)$

and $\|M_{\nu_f}\|_{L^p} \leq A_p \|f\|_{L^p}$

constant depending on p

see the book of Stein

Since this last inequality holds true for all the compact sets contained in E, the conclusion follows. $\square$

Remark 15. *Usually the Hardy-Littlewood theorem is stated in a slightly different way. In particular let $f \in L^1(\mathbb{R}^d)$. Denote by M_f the function*

$$M_f(x) = \sup_{r>0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f| \, d\lambda.$$

The result shown here above implies that, for all $\alpha > 0$,

$$\lambda(\{x \in \mathbb{R}^d \mid M_f(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \cdot \|f\|_{L^1}.$$

It is possible to prove also that, if $f \in L^p(\mathbb{R}^d)$, with $1 < p \leq +\infty$, then $M_f \in L^p(\mathbb{R}^d)$ and

$$\|M_f\|_{L^p} \leq A_p \|f\|_{L^p},$$

where A_p depends only on p and d (see [8, Ch. 1]).

7.2 Lebesgue's points

Definition 19. *Let $f \in L^1_{loc}(\mathbb{R}^d)$ (this means that, for all K compact sets in $\mathbb{R}^d$, $\chi_K \cdot f \in L^1(\mathbb{R}^d)$). Let $x \in \mathbb{R}^d$. x is said to be a Lebesgue's point for f if*

$$f(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \, dy,$$

(from now on, given $A \in \mathcal{B}$, $|A|$ will denote the Lebesgue measure of A).

Theorem 25. *Let $f \in L^1(\mathbb{R}^d)$.*
Then, for almost all $x \in \mathbb{R}^d$,

$$\lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| \, dy = 0.$$

Proof. Let $g \in C^0_0(\mathbb{R}^d)$. Then

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| \, dy$$
$$\leq \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - g(y)| \, dy + \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - f(x)| \, dy.$$

Denoting by M_{f-g} the maximal function of $f - g$, i. e.

$$M_{f-g}(x) = \sup_{r>0} \frac{1}{|(B(x,r)|} \int_{B(x,r)} |f(y) - g(y)| \, dy,$$

we have

$$\begin{aligned} &\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)| dy \\ &\qquad \leq M_{f-g}(x) + \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - f(x)| dy. \end{aligned} \tag{13}$$

We consider, on both sides of (13), the $\limsup_{r\to 0^+}$. We have

$$T(x) = \limsup_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\, dy$$
$$\leq M_{f-g}(x) + \limsup_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - f(x)|\, dy.$$

Remarking that the function $y \mapsto |g(y) - f(x)|$ is a continuous function, we obtain that

$$\limsup_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |g(y) - f(x)|\, dy = |g(x) - f(x)|,$$

and, finally,

$$T(x) \leq M_{f-g}(x) + |g(x) - f(x)|.$$

Take now $\varepsilon > 0$ and consider

$$\{x \in \mathbb{R}^d \mid T(x) > 2\varepsilon\} \subseteq \{x \in \mathbb{R}^d \mid M_{f-g}(x) > \varepsilon\} \cup \{x \in \mathbb{R}^d \mid |f(x) - g(x)| > \varepsilon\}.$$

From the Hardy-Littlewood theorem we know that

$$\lambda(\{x \in \mathbb{R}^d \mid M_{f-g}(x) > \varepsilon\}) \leq 3^d \cdot \frac{1}{\varepsilon} \cdot \|f - g\|_{L^1}$$

and, from a direct calculation,

$$\lambda(\{x \in \mathbb{R}^d \mid |f(x) - g(x)| > \varepsilon\}) \leq \frac{1}{\varepsilon} \cdot \|f - g\|_{L^1}.$$

Hence

$$\lambda(\{x \in \mathbb{R}^d \mid T(x) > 2\varepsilon\}) \leq (3^d + 1)\frac{1}{\varepsilon}\|f - g\|_{L^1}. \tag{14}$$

It is sufficient to consider a sequence $(g_n)_n$ in C_0^0 such that, for all n, $\|f - g_n\|_{L^1} < 1/n$ (see Theorem 1 of Lesson 8), obtaining, from (14), that, for all $\varepsilon > 0$, we have

$$\lambda(\{x \in \mathbb{R}^d \mid T(x) > 2\varepsilon\}) = 0.$$

In conclusion, for almost every $x \in \mathbb{R}^d$,

$$\lim_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0.$$

□

Corollary 9. *Let $f \in L^1_{loc}(\mathbb{R}^d)$.*
Then, for almost every $x \in \mathbb{R}^d$,

$$\lim_{r\to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x).$$

Remark 16. *Let ν be a complex measure on $\mathcal{B}(\mathbb{R}^d)$. Suppose that $\nu \ll \lambda$, where λ is the Lebesgue measure. From the Radon-Nikodym theorem we know that there exists $f_0 \in L^1_\lambda(\mathbb{R}^d)$ such that, for all $E \in \mathcal{B}$,*

$$\nu(E) = \int_E f_0\, d\lambda.$$

Lebesgue's points

def let $f \in L^1_{loc}(\mathbb{R}^d)$ ($f \in L^1_{loc}(\mathbb{R}^d)$ means that f is measurable and $\forall K$ compact in $\mathbb{R}^d$ $f \cdot \chi_K \in L^1(\mathbb{R}^d)$)

$x \in \mathbb{R}^d$

x is a Lebesgue's point for f if

$$\lim_{r \to 0^+} \underbrace{\frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y)\,dy}_{\text{integral mean on } B(x,r)} = f(x)$$

Theorem let $f \in L^1_{loc}(\mathbb{R}^d)$. then for almost all $x \in \mathbb{R}^d$

$$\lim_{r \to 0^+} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\,dy = 0$$

Corollary $f \in L^1_{loc}(\mathbb{R}^d)$

then almost all $x \in \mathbb{R}$ are Lebesgue's points proof. I show the theorem for $f \in L^1$ (and not in L^1_{loc})

take $g \in \mathcal{C}_0(\mathbb{R}^d)$ (= continuous functions with compact support)

(g is 0 outside a compact set of $\mathbb{R}^d$)

I compute

$$\frac{1}{\lambda(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\,dy$$

$$\leq \underbrace{\frac{1}{\lambda(B(x,r))}\int_{B(x,r)} |f(y)-g(y)|\,dy}_{\underbrace{\frac{1}{\lambda(B(x,r))}\,\nu_{|f-g|}(B(x,r))}} + \frac{1}{\lambda(B(x,r))}\int_{B(x,r)} |g(y)-f(x)|\,dy$$

$$\left(\nu_{|f-g|}(A) = \int_A |f-g|\right)$$

$$\leq M_{\nu_{|f-g|}}(x) + \frac{1}{\lambda(B(x,r))}\int_{B(x,r)} |g(y)-f(x)|\,dy$$

$$T(x) = \limsup_{r\to 0^+} \frac{1}{\lambda(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\,dy$$

$$T(x) \leq M_{\nu_{|f-g|}}(x) + \limsup_{r\to 0^+} \frac{1}{\lambda(B(x,r))}\int_{B(x,r)} \underbrace{|g(y)-f(x)|}\,dy$$

this function of y is continuous function

so that

$$\boxed{T(x) \leq M_{\nu_{|f-g|}} + |g(x)-f(x)|}$$

int. mean for a cont. funct.

To be continued....

From Corollary 9 we have also that, for almost every $x \in \mathbb{R}^d$,

$$f_0(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f_0(y)\, dy = \lim_{r \to 0^+} \frac{\nu(B(x,r))}{\lambda(B(x,r))} = \frac{d\nu}{d\lambda}(x).$$

Hence, if $\nu \ll \lambda$, then, for almost every $x \in \mathbb{R}^d$, ν possesses a finite symmetric derivative with respect to λ and the value of the symmetric derivative is exactly the value of the Radon-Nikodym density function, i. e., for all $E \in \mathcal{B}$,

$$\nu(E) = \int_E \frac{d\nu}{d\lambda}(x)\, d\lambda.$$

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