Corollary 9. *Let $f \in L^1_{loc}(\mathbb{R}^d)$.*

Then, for almost every $x \in \mathbb{R}^d$,

$$\lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x).$$

Remark 16. *Let ν be a complex measure on $\mathcal{B}(\mathbb{R}^d)$. Suppose that $\nu \ll \lambda$, where λ is the Lebesgue measure. From the Radon-Nikodym theorem we know that there exists $f_0 \in L^1_\lambda(\mathbb{R}^d)$ such that, for all $E \in \mathcal{B}$,*

$$\nu(E) = \int_E f_0 \, d\lambda.$$

From Corollary 9 we have also that, for almost every $x \in \mathbb{R}^d$,

$$f_0(x) = \lim_{r \to 0^+} \frac{1}{|B(x,r)|} \int_{B(x,r)} f_0(y)\, dy = \lim_{r \to 0^+} \frac{\nu(B(x,r))}{\lambda(B(x,r))} = \frac{d\nu}{d\lambda}(x).$$

Hence, if $\nu \ll \lambda$, then, for almost every $x \in \mathbb{R}^d$, ν possesses a finite symmetric derivative with respect to λ and the value of the symmetric derivative is exactly the value of the Radon-Nikodym density function, i. e., for all $E \in \mathcal{B}$,

$$\nu(E) = \int_E \frac{d\nu}{d\lambda}(x)\, d\lambda.$$

8 Lesson 9 – March 29$^{\text{th}}$, 2021

8.1 Preliminary results (to distribution theory)

8.1.1 $C_0(\Omega)$ is dense in $L^1(\Omega)$

The following density result is considered (by H. Brezis) "un résultat d'intégration qu'il faut absolument connaître".

Theorem 26 (Th. IV.3 in [1])**.** *Let Ω be an open set in $\mathbb{R}^n$. Let $f \in L^1(\Omega)$. Let $\varepsilon > 0$.*

Then there exists $\varphi \in C_0(\Omega)$ such that

$$\|f - \varphi\|_{L^1(\Omega)} < \varepsilon,$$

i. e. $C_0(\Omega)$ is dense in $L^1(\Omega)$, where $C_0(\Omega)$ denotes the space of continuous functions φ such that the closure of the set $\{x \in \Omega \mid \varphi(x) \neq 0\}$, i. e. the support of φ, is a compact set in Ω.

8.1.2 $C_0(\Omega)$ is dense in $L^p(\Omega)$, for all $1 \leq p < +\infty$

The density result of the previous paragraph can be extended to L^p, for all $1 \leq p < +\infty$.

Lemma 10 (Lemma IV.2 in [1])**.** *Let $f \in L^1_{loc}(\Omega)$. Suppose that, for all $\varphi \in C_0(\Omega)$,*

$$\int_\Omega f\varphi = 0.$$

Then $f = 0$.

Lesson 9 (March 29^{th})

$$M_\nu(x) = \sup_{r>0} \frac{|\nu|(B(x,r))}{\lambda(B(x,r))}$$

ν is a complex measure on $\mathcal{B}$ Borelian sets ($|\nu|(\mathbb{R}^d) < +\infty$)

- $M_\nu : \mathbb{R}^d \to [0,+\infty]$ is lower semicontinuous

$\forall \alpha \geq 0$, $E = \{x \in \mathbb{R}^d : M_\nu(x) > \alpha\}$ is open

- (H-L maximal th.)

$$\forall \alpha > 0, \quad \lambda(\{x \in \mathbb{R}^d : M_\nu(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \cdot |\nu|(\mathbb{R}^d)$$

(d = dimension of $\mathbb{R}^d$)

Rem. This is true in particular for $\nu = \nu_f$

$$\nu_f(A) = \int_A f \, d\lambda \qquad f \in L^1_\lambda(\mathbb{R}^d)$$

$$\lambda(\{M_{\nu_f}(x) > \alpha\}) \leq 3^d \cdot \frac{1}{\alpha} \cdot \|f\|_{L^1}$$

Lebesgue point $\quad f \in L^1_{loc}(\mathbb{R}^d)$

x_0 is a Lebesgue's point if

$$\lim_{r \to 0^+} \frac{1}{\lambda(B(x_0,r))} \int_{B(x_0,r)} f(y)\,dy = f(x_0)$$

Th. $f \in L^1_{loc}(\mathbb{R}^d)$

then for almost all $x \in \mathbb{R}^d$

$$\lim_{r \to 0^+} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\,dy = 0$$

Corollary $\quad f \in L^1_{loc}(\mathbb{R}^d) \Rightarrow$ almost all $x \in \mathbb{R}^d$ are Lebesgue's points proof. not restrictive if $f \in L^1(\mathbb{R}^d)$

take $g \in \mathcal{C}_0(\mathbb{R}^d) = \{$ cont. functions with compact support $\}$

$$T(x) = \limsup_{r \to 0^+} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(x) - f(y)|\, dy$$

We arrived on

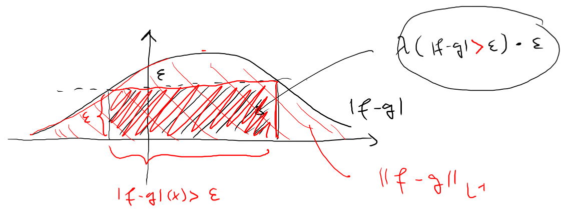
$$T(x) \leq M_{\nu_{|f-g|}}(x) + |f(x) - g(x)|$$

Fix $\varepsilon > 0$

$$\{x \in \mathbb{R}^d : T(x) > 2\varepsilon\} \subseteq \{x \in \mathbb{R}^d : M_{\nu_{|f-g|}}(x) > \varepsilon\} \cup \{x \in \mathbb{R}^d : |f(x) - g(x)| > \varepsilon\}$$

$$\lambda(\{x \in \mathbb{R}^d : M_{\nu_{|f-g|}}(x) > \varepsilon\}) \leq 3^d \cdot \frac{1}{\varepsilon} \cdot \|f-g\|_{L^1} \quad \leftarrow \text{H. L.}$$

$$\lambda(\{x \in \mathbb{R}^d : |f(x) - g(x)| > \varepsilon\}) \leq \frac{1}{\varepsilon} \|f-g\|_{L^1} \quad \uparrow ?$$



Remember (we will go back to it in the future)

$\mathcal{C}_0(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$

so for all $n \in \mathbb{N}$, $\exists\, g_n \in L^1$ s.t. $\|f - g_n\|_{L^1} < \frac{1}{n}$

then $\lambda(\underbrace{\{x \in \mathbb{R}^d : M_{\nu_{|f-g_n|}}(x) > \varepsilon\} \cup \{x \in \mathbb{R}^d : |f(x) - g_n(x)| > \varepsilon\}}_{A_n}) \leq (3^d + 1)\frac{1}{\varepsilon} \cdot \frac{1}{n}$

$\forall n, \exists g_n$

$$\{x \in \mathbb{R}^d : T(x) > 2\varepsilon\} \subseteq \underbrace{\bigcap_n A_n}$$

$$\lambda(\bigcap_n A_n) = 0$$

$$\Rightarrow \lambda(\{x \in \mathbb{R}^d : T(x) > 2\varepsilon\}) = 0$$

this is true for all $\varepsilon > 0$

$$\lambda(\{x \in \mathbb{R}^d : T(x) > 0\}) = 0$$

i.e.

$$\limsup_{r \to 0^+} \underbrace{\frac{1}{\lambda(B(x,r))} \int_{B(x,r)} |f(x) - f(y)|\, dy}_{\geq 0} = 0$$

$\| \quad \lim_{r \to 0^+}$

QED

Remark: take ν a complex measure on $\mathcal{B}$ = Borelian sets on $\mathbb{R}^d$

suppose $\nu << \lambda$
absolutely continuous w.r.t. Lebesgue's.

Apply Radon-Nikodym

$\exists\, f_0 \in L^1_\lambda(\mathbb{R}^d)$ s.t. $\forall A \in \mathcal{B}$,

(density of ν)

$$\nu(A) = \int_A f_0(y)\, dy$$

but $f_0 \in L^1_\lambda(\mathbb{R})$ so also the theorem on Lebesgue points holds

so that

for almost all x

$$\lim_{r \to 0^+} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f_0(y)\, dy = f_0(x)$$

$$\lim_{r \to 0^+} \frac{\nu(B(x_0,r))}{\lambda(B(x_0,r))} = \text{symmetric derivative of } \nu \text{ w.r.t. } \lambda$$

$$= \frac{d\nu}{d\lambda}(x_0)$$

final formula

if $\nu << \lambda$

$$\nu(A) = \int_A \frac{d\nu}{d\lambda}(y)\, dy$$

the symmetric derivative is the density of Radon-Nikodym

Preliminary results (to distribution theory)

1) $\mathcal{C}_0(\Omega)$ is dense in $L^1(\Omega)$

Ω is an open set of $\mathbb{R}^d$

$\mathcal{C}_0(\Omega) = \{ f$ continuous having compact support in $\Omega \}$

($f : \Omega \to \mathbb{R}$ (or $\mathbb{C}$), continuous.

$\text{supp } f = \overline{\{ x \in \Omega : f(x) \neq 0 \}}$ ↖ closure in Ω

remark that if $f \in \mathcal{C}_0(\Omega)$, considering

$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \notin \Omega \end{cases}$$

then $\bar{f} \in \mathcal{C}_0(\mathbb{R}^d)$

in general $f \in \mathcal{C}(\Omega)$ and $\bar{f} \notin \mathcal{C}(\mathbb{R}^d)$!!

from now on, if $f \in \mathcal{C}_0(\Omega)$, we use f also for the function extended with 0 outside Ω.)

Th. Ω open set of $\mathbb{R}^d$

$\mathcal{C}_0(\Omega)$ is dense in $L^1(\Omega)$.

2) $\mathscr{C}_0(\Omega)$ is dense in $L^p(\Omega)$
for all $1 \le p < +\infty$ (+∞ excluded)

Lemma Let $f \in L^1_{loc}(\Omega)$
suppose that $\forall \varphi \in \mathscr{C}_0(\Omega)$, $\int_\Omega f\varphi = 0$
then $f = 0$.

also in Zagatti's ?

Hint. (See Brezis ch. 4)
use Uryshon lemma and other considerations

Theorem let $1 \le p < +\infty$, Ω open set in $\mathbb{R}^d$.
then $\mathscr{C}_0(\Omega)$ is dense in $L^p(\Omega)$

proof. if $p = 1$ is the point 1) of today's lesson.

let $1 < p < +\infty$

Consider $\mathscr{C}_0(\Omega)$ as a subspace of $L^p(\Omega)$

Let $\Phi \in (L^p(\Omega))'$

suppose that $\Phi_{|\mathscr{C}_0(\Omega)} \equiv 0$

if I deduce that $\Phi \equiv 0$ on $L^p(\Omega)$
then $\mathscr{C}_0(\Omega)$ is dense thank to a corollary to Hahn-Banach.

$$\Phi_{|\mathscr{C}_0(\Omega)} = 0 \iff \Phi(\varphi) = 0 \quad \forall \varphi \in \mathscr{C}_0(\Omega)$$

$1 < p < +\infty$

I use Riesz's theorem $\Phi \in (L^p)' \iff$
$\exists!\, g \in L^{p'}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$
s.t.
$$\forall v \in L^p \quad \Phi(v) = \int_\Omega g v$$

so $\exists\, g \in L^{p'} (\subseteq L^1_{loc}(\Omega))$ s.t. $\int_\Omega g\varphi = \Phi(\varphi) = 0$ $\forall \varphi \in \mathscr{C}_0$

From the lemma $g \equiv 0$

so that $\Phi \equiv 0$ $\Rightarrow$ the conclusion from H-B.

QED

Proof. Let us suppose that $f \in L^1(\Omega)$ and $|\Omega| < +\infty$.

Since $C_0(\Omega)$ is dense in $L^1(\Omega)$, then, for all $\varepsilon > 0$, there exists $f_\varepsilon \in C_0(\Omega)$ such that

$$\|f - f_\varepsilon\|_{L^1(\Omega)} < \varepsilon.$$

Consequently, for all $\varphi \in C_0(\Omega)$,

$$|\int_\Omega f_\varepsilon \varphi| = |\int_\Omega (f_\varepsilon - f)\varphi| \leq \|f - f_\varepsilon\|_{L^1(\Omega)} \|\varphi\|_{L^\infty(\Omega)} < \varepsilon \|\varphi\|_{L^\infty(\Omega)}. \tag{15}$$

Consider

$$K_1 = \{x \in \Omega \mid f_\varepsilon(x) \geq \varepsilon\}, \qquad K_2 = \{x \in \Omega \mid f_\varepsilon(x) \leq -\varepsilon\}$$

and $K = K_1 \cup K_2$. K_1, K_2 and K are compact sets in Ω. We use Uryshon's Lemma to construct $u_\varepsilon \in C_0(\Omega)$ such that

$$|u_\varepsilon(x)| \leq 1 \quad \text{for all} \quad x \in \Omega \qquad \text{and} \qquad \begin{cases} u_\varepsilon(x) = 1 & \text{on} \quad K_1, \\ u_\varepsilon(x) = -1 & \text{on} \quad K_2. \end{cases}$$

We have

$$\int_\Omega |f| \leq \underbrace{\int_\Omega |f - f_\varepsilon|}_{\leq \varepsilon} + \int_\Omega |f_\varepsilon| \leq \varepsilon + \int_{\Omega \setminus K} |f_\varepsilon| + \int_K |f_\varepsilon|.$$

Remark now that

$$\int_K |f_\varepsilon| = \int_K f_\varepsilon u_\varepsilon = \int_\Omega f_\varepsilon u_\varepsilon - \int_{\Omega \setminus K} f_\varepsilon u_\varepsilon$$

and

$$|\int_\Omega f_\varepsilon u_\varepsilon| \leq \varepsilon \|u_\varepsilon\|_{L^\infty} \leq \varepsilon, \qquad \text{as a consequence of (15),}$$

$$|\int_{\Omega \setminus K} f_\varepsilon u_\varepsilon| \leq \int_{\Omega \setminus K} |f_\varepsilon| \leq \varepsilon \cdot |\Omega \setminus K|, \qquad \text{since, on } \Omega \setminus K \text{, we have } |f_\varepsilon| \leq \varepsilon,$$

so that

$$\int_K |f_\varepsilon| \leq \underbrace{|\int_\Omega f_\varepsilon u_\varepsilon|}_{\leq \varepsilon} + \underbrace{|\int_{\Omega \setminus K} f_\varepsilon u_\varepsilon|}_{\leq \varepsilon \cdot |\Omega \setminus K|} \leq \varepsilon(1 + |\Omega \setminus K|).$$

Finally

$$\int_\Omega |f| \leq \underbrace{\int_\Omega |f - f_\varepsilon|}_{\leq \varepsilon} + \int_\Omega |f_\varepsilon| \leq \varepsilon + \underbrace{\int_{\Omega \setminus K} |f_\varepsilon|}_{\varepsilon \cdot |\Omega \setminus K|} + \underbrace{\int_K |f_\varepsilon|}_{\varepsilon(1 + |\Omega \setminus K|)} \leq 2\varepsilon(1 + |\Omega|).$$

This last inequality implies that $\int_\Omega |f| = 0$ and consequently $f = 0$.

Suppose now $f \in L^1_{loc}$ and Ω open in $\mathbb{R}^n$. Consider

$$\Omega_n = B(0, n) \cap \{x \in \Omega \mid \operatorname{dist}(x, \mathcal{C}\Omega) > \frac{1}{n}\}.$$

From what we have already proved, we deduce that, for all n,

$$f \cdot \chi_{\Omega_n} = 0,$$

and this conclude the proof. $\square$

Theorem 27 (Th IV.12 in [1])**.** *$C_0(\Omega)$ is dense in $L^p(\Omega)$, for all $1 \leq p < +\infty$.*

Proof. This result, in the case $p = 1$, is already known. Let $1 < p < +\infty$. We know that a consequence of the Hahn-Banach theorem is the following: let W a subspace of a normed space $V$and suppose that, for all $\Phi \in V'$, $\Phi(W) = 0$ implies $\Phi = 0$, then W is a dense subspace of V. Consider $\Phi \in (L^p(\Omega))'$. From Riesz's theorem we have that there exists $g \in L^{p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$, such that

$$\Phi(\varphi) = \int_\Omega g\varphi$$

Suppose that $\Phi(\varphi) = 0$ for all $\varphi \in C_0(\Omega)$, i. e. $\int_\Omega g\varphi = 0$ for all $\varphi \in C_0(\Omega)$. From the previous lemma we have that $g = 0$, i. e. $\Phi = 0$. As a consequence $C_0(\Omega)$ is dense in $L^p(\Omega)$. $\square$

8.1.3 Convolution of functions

We collect here some (supposed) known results on convolution (see [1, Ch. IV.4]).

Theorem 28 (Th. IV.15 in [1])**.** *Let $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, with $1 \leq p \leq +\infty$.*

Then, for almost every $x \in \mathbb{R}^n$, the function

$$y \mapsto f(x-y)g(y) \qquad \text{is in} \qquad L^1(\mathbb{R}^n)$$

and setting

$$f * g\,(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$$

we have $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}.$$

More generally, let $1 \leq p,\ q,\ r \leq +\infty$, with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. Let $f \in L^r(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$.

Then

$$f * g \in L^q(\mathbb{R}^n) \qquad \text{and} \qquad \|f * g\|_{L^q} \leq \|f\|_{L^r}\|g\|_{L^p} \qquad \text{(Young inequality).}$$

Definition 20. *Let f be a continuous function defined on Ω, open set of $\mathbb{R}^n$. We call support of f the closure, in Ω, of the set $\{x \in \Omega \mid f(x) \neq 0\}$.*

Let f be a $L^1_{loc}(\Omega)$ function. Consider W, the set of points of Ω, having an open neighborhood U in Ω, such that f is identically equal to 0 on U. We call support of f the complementary set of W in Ω.

The support of f in Ω is the largest relatively closed set in Ω outside of which f is identically equal to 0.

3) convolution

Th. let $f \in L^1(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$ $1 \le p \le +\infty$

then for almost all $x \in \mathbb{R}$

$y \longmapsto f(x-y)\,g(y)$ is in L^1

and denoting by $f*g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$

then $f*g \in L^p(\mathbb{R}^d)$ and

$$\|f*g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}$$

(more generally (Young inequality)

if $1 \le p, q, r \le +\infty$ s.t. $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$

then $f \in L^r$, $g \in L^p$

then $f*g \in L^q$ and $\|f*g\|_{L^q} \le \|f\|_{L^r} \|g\|_{L^p}$)

def. let $f \in L^1_{loc}(\Omega)$
consider the set $\tilde{\Omega} = \{ y \in \Omega : \exists\, \varepsilon > 0$ s.t. $f_{|B(y,\varepsilon)}$ is identically $0 \}$

then we define $\Omega \setminus \tilde{\Omega}$ as the support of f

remark that also for $f \in L^1_{loc}(\Omega)$ supp f is closed set

Exercise: take $f \in \mathcal{C}_0(\Omega)$ then

verify that the support as continuous function
the support as L^1_{loc} function
are the same.

Theorem $f \in L^1(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$

then $\mathrm{supp}\,(f*g) \subseteq \overline{\mathrm{supp}\, f + \mathrm{supp}\, g}$

Theorem $f \in \mathcal{C}_0(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$

then $f*g \in \mathcal{C}(\mathbb{R}^d)$

in such a situation $f*g$ still exists

and $f \in \mathcal{C}_0^m(\mathbb{R}^m)$ and $g \in L^1_{loc}(\mathbb{R}^d)$

then $f*g \in \mathcal{C}^m(\mathbb{R}^d)$

and $$\frac{\partial}{\partial x_j}(f*g) = \frac{\partial f}{\partial x_j} * g$$

Theorem 29 (Prop. IV.18 in [1]). *Let $f \in L^1(\mathbb{R}^n)$, $g \in L^p(\mathbb{R}^n)$, with $1 \le p \le +\infty$. Then*

$$\mathrm{Supp}\,(f * g) \subseteq \overline{\mathrm{Supp}\, f + \mathrm{Supp}\, g}$$

Remark 17. *Let $f \in L^1(\mathbb{R}^n)$ with compact support (i. e. f is identically equal to 0 outside a compact). Let $g \in L^p_{loc}(\mathbb{R}^n)$. Then it is possible to define $f * g$ in the usual way and we have that $f * g \in L^p(\mathbb{R}^n)$.*

Theorem 30 (Prop. IV.20 in [1]). *Let $f \in C_0(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$.*
Then $f * g \in C(\mathbb{R}^n)$.

Let $f \in C_0^m(\mathbb{R}^n)$, with $m \ge 1$, and $g \in L^1_{loc}(\mathbb{R}^n)$.
Then

$$f * g \in C^m(\mathbb{R}^n) \qquad \text{and} \qquad \frac{\partial}{\partial x_j}(f * g) = \frac{\partial f}{\partial x_j} * g.$$

8.1.4 Test functions and mollifiers

We collect here some notions on test functions and mollifiers (see [3, Ch. 1.2]).

Definition 21. *We set*

$$C_0(\Omega) = \{\text{continuous functions with compact support contained in } \Omega\},$$

for $m \in \mathbb{N}$,

$$C_0^m(\Omega) = C_0(\Omega) \cap C^m(\Omega),$$

and, finally,

$$\mathcal{D}(\Omega) = C_0^\infty(\Omega) = \bigcap_m C_0^m(\Omega).$$

The elements of $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ are called test functions.

Example 2. *Let*

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \le 0. \end{cases}$$

It is possible to prove that $f \in C^\infty(\mathbb{R})$ and $f^{(j)}(t) = 0$ for all j and for all $t \le 0$. The function

$$u : \mathbb{R}^n \to \mathbb{R}, \qquad u(x) = f(1 - |x|^2),$$

is a test function, with $\mathrm{Supp}\, u = \overline{B(0,1)}$.

Definition 22. *Let $\rho \in \mathcal{D}(\mathbb{R}^d)$, $\rho \ge 0$, $\mathrm{Supp}\,\rho \subseteq \overline{B(0,1)}$ and $\int_{\mathbb{R}^d} \rho(x)\, dx = 1$. The set*

$$\{\rho_\varepsilon, \;\big|\; \varepsilon \in]0,1], \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\, \rho(\frac{x}{\varepsilon})\} = (\rho_\varepsilon)_{\varepsilon \in]0,1]},$$

is called mollifier (or also family of mollifiers). Similarly we will call mollifier (or family of mollifiers) the sequence

$$(\rho_n)_n \qquad \text{with} \qquad \rho_n(x) = n^d \rho(nx).$$

Theorem 31 (Th. 1.2.1 in [3]). *Let $(\rho_\varepsilon)_\varepsilon$ be a mollifier.*

4) Mollifiers and Test functions.

def. Let Ω open set in $\mathbb{R}^d$

we define $\mathcal{C}_0^\infty(\Omega) = \mathcal{C}^\infty(\Omega) \cap \mathcal{C}_0(\Omega)$

$= \mathcal{D}(\Omega)$ (notation of Laurent Schwartz)

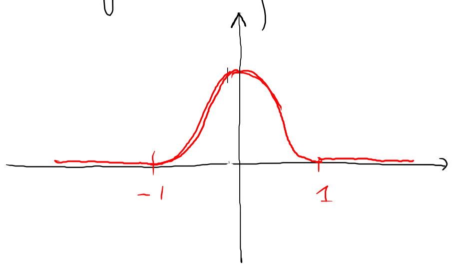
test functions.

Example consider $f : \mathbb{R} \to \mathbb{R}$ $\quad f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{for } t>0 \\ 0 & \text{for } t=0 \end{cases}$

verify that $f \in \mathcal{C}^\infty$

in particular $f^{(j)}(0) = 0 \quad \forall j$

define $\rho : \mathbb{R}^d \to \mathbb{R}$ $\quad \rho(x) = f(1 - |x|^2)$



$\text{supp}(\rho) \subseteq \overline{B(0,1)}$

$\rho(x) \geq 0$

$\boxed{\rho \in \mathcal{D}(\mathbb{R}^d)}$

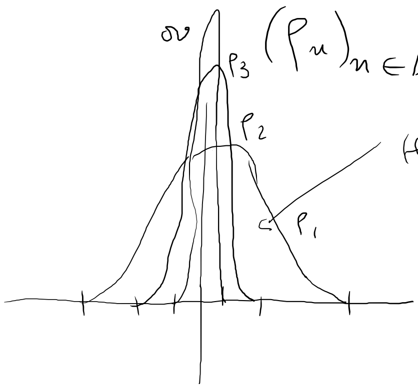
def consider $\rho \in \mathcal{D}(\mathbb{R}^d)$

$\rho(x) \geq 0, \quad \text{supp}\, \rho \subseteq \overline{B(0,1)}, \quad \int_{\mathbb{R}^d} \rho(x)\,dx = 1$

I call mollifier

the family $(\rho_\varepsilon)_{\varepsilon \in]0,1]}$ $\quad \rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$

or $(\rho_n)_{n \in \mathbb{N}}$ $\quad \rho_n(x) = n^d \rho(nx)$



Theorem let $(\rho_\varepsilon)_{\varepsilon \in (0,1]}$ a family of mollifiers, (let Ω an open set in $\mathbb{R}^d$)

i) if $u \in L^1(\Omega)$ such that $\text{supp}\, u$ is compact set in Ω,

then, for $\varepsilon \in (0, \varepsilon_0)$, $\quad \rho_\varepsilon * \bar{u} \in \mathcal{D}(\Omega)$

(there exists $\varepsilon_0 > 0$) $\quad$ where $\bar{u}(x) = \begin{cases} u(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$

ii) if $u \in \mathcal{C}_0(\Omega)$

then $\rho_\varepsilon * u \to u$ uniformly.

iii) if $u \in L^p(\Omega)$ with $1 \leq p < +\infty$

then $\rho_\varepsilon * \bar{u} \to u$ in $L^p(\Omega)$

proof i) take $\varepsilon_0 = \text{dist}(\text{supp}\, u, \partial\Omega)$

> 0 since u has compact support in Ω

then $\text{supp}\, \rho_\varepsilon * \bar{u} \subseteq \text{supp}\, u + B(0,\varepsilon)$

is compact of Ω

the regularity $\mathcal{C}^\infty$ of $\rho_\varepsilon * \bar{u}$ is consequence of the property of convolution.

i) Let $u \in L^1(\Omega)$, with $u = 0$ outside a compact set of Ω.
Then there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, $\rho_\varepsilon * u \in C_0^\infty(\Omega)$.

ii) Let $u \in C_0(\Omega)$.
Then, for ε going to 0^+, $\rho_\varepsilon * u$ converges uniformly to u.

iii) Let $u \in L^p(\Omega)$, with $1 \le p < +\infty$. Let

$$\bar{u}(x) = \begin{cases} u(x) & \text{for} \quad x \in \Omega, \\ 0 & \text{for} \quad x \notin \Omega. \end{cases}$$

Then, for ε going to 0^+, $\rho_\varepsilon * \bar{u}$ converges to u in $L^p(\Omega)$.

Proof. i) Denote by K the compact set of Ω outside of which the function u is identically 0. Take $\varepsilon_0 > 0$ less than the distance between K and the border of Ω. Theorem 4 and Theorem 5 give the conclusion.

ii) Let $\varepsilon_0 > 0$ as in the previous point, and let $0 < \varepsilon < \varepsilon_0$. Then

$$\rho_\varepsilon * u\,(x) - u(x) = \int_{|y| \le \varepsilon} \rho_\varepsilon(y)(u(x-y) - u(x))\,dy.$$

Consider now that u is uniformly continuous, so that for all $r > 0$ there exists $\delta > 0$ such that, if $|x_1 - x_2| < \delta$ then $|u(x_1) - u(x_2)| < r$. Consequently, if $\varepsilon < \delta$, for all $x \in \Omega$,

$$|\rho_\varepsilon * u\,(x) - u(x)| \le \int_{|y| \le \varepsilon} \rho_\varepsilon(y)|u(x-y) - u(x)|\,dy \le \int_{|y| \le \varepsilon} \rho_\varepsilon(y) r\,dy = r$$

and the conclusion follows.

iii) We know that $C_0(\Omega)$ is dense in $L^p(\Omega)$ (recall that $1 \le p < +\infty$). Fix $\delta > 0$ and consider $w \in C_0(\Omega)$ such that $\|u - w\|_{L^p(\Omega)} < \delta$. We have

$$\begin{aligned} &\|(\rho_\varepsilon * \bar{u}) - u\|_{L^p(\Omega)} \\ &\quad \le \|(\rho_\varepsilon * \bar{u}) - \bar{u}\|_{L^p(\mathbb{R}^n)} \\ &\quad \le \|(\rho_\varepsilon * \bar{u}) - (\rho_\varepsilon * w)\|_{L^p(\mathbb{R}^n)} + \|(\rho_\varepsilon * w) - w\|_{L^p(\mathbb{R}^n)} + \|w - u\|_{L^p(\Omega)}. \end{aligned}$$

We consider now the fact that

$$\|(\rho_\varepsilon * \bar{u}) - (\rho_\varepsilon * w)\|_{L^p(\mathbb{R}^n)} = \|\rho_\varepsilon * (\bar{u} - w)\|_{L^p(\mathbb{R}^n)} \le \|\rho_\varepsilon\|_{L^1} \|u - w\|_{L^p(\Omega)} \le \delta,$$

and

$$\|w - u\|_{L^p(\Omega)} \le \delta.$$

Consequently

$$\|(\rho_\varepsilon * \bar{u}) - u\|_{L^p(\Omega)} \le \|(\rho_\varepsilon * w) - w\|_{L^p(\mathbb{R}^n)} + 2\delta.$$

From the point ii) we know that $\rho_\varepsilon * w$ is converging uniformly on Ω to w and both $\rho_\varepsilon * w$ and w are $C_0(\Omega)$ functions, so that $\rho_\varepsilon * w$ is converging to w also in $L^p(\mathbb{R}^n)$. This means that, if ε is sufficiently small,

$$\|(\rho_\varepsilon * \bar{u}) - u\|_{L^p(\Omega)} \le 3\delta,$$

and the proof is complete. □

Remark 18. *Convolution with a mollifier is a good way to construct a* C_0^∞ *function which value is 1 in a neighborhood of a certain compact* K*. Let's show how to do it.*

Let K *be a compact set in* $\mathbb{R}^n$*. Consider the covering* $\{B(x, \varepsilon_0) \mid x \in K\}$ *and extract a finite subcovering*

$$B(x_1, \varepsilon_0),\ B(x_2, \varepsilon_0), \ldots, B(x_N, \varepsilon_0).$$

Define

$$K_1 = \bigcup_{j=1}^{N} \overline{B(x_j, 2\varepsilon_0)}$$

and finally consider $\rho_\varepsilon * \chi_{K_1}$*, with* $\varepsilon < \varepsilon_0$*. We let as an exercise to verify that* $\rho_\varepsilon * \chi_{K_1}$ *is a* C_0^∞ *and that its value is* 1 *inside each ball* $B(x_j, \varepsilon_0)$*.*

We end this paragraph with a refinement of the previous density results.

Lemma 11. *Let* $f \in L^1_{loc}(\Omega)$*. Suppose that for all* $\varphi \in C_0^\infty(\Omega)$*,* $\int_\Omega f\varphi = 0$*.*
Then $f = 0$*.*

Proof. Suppose first that $f \in L^1(\Omega)$. Let $\psi \in C_0(\Omega)$. Let $(\rho_n)_n$ be a mollifier. Consider $\varphi_n = \rho_n * \psi$. We have that, for all n, $\varphi_n \in C_0^\infty$ and φ_n converges uniformly to ψ. Remark that

$$|\varphi_n(x)| = |\int_{\mathbb{R}^n} \rho_n(y)\psi(x-y)\,dy| \leq \max|\psi| \int_{\mathbb{R}^n} |\rho_n(y)|\,dy \leq \max|\psi|.$$

Then

$$f(x)\varphi_n(x) \xrightarrow{n} f(x)\psi(x) \qquad \text{almost everywhere,}$$

and

$$|f(x)\varphi_n(x)| \leq \max|\psi||f(x)|.$$

We can apply the dominated convergence theorem and we have

$$\int_\Omega f(x)\varphi_n(x)\,dx \xrightarrow{n} \int_\Omega f(x)\psi(x)\,dx,$$

but we know that, for all n, $\int_\Omega f(x)\varphi_n(x)\,dx = 0$, so that $\int_\Omega f(x)\psi(x)\,dx$. The conclusion is a consequence of Lemma 1.

Let now f be in $L^1_{loc}(\Omega)$. The above part of the proof guarantees that, for all compact set K, the function $f \cdot \chi_K$ is identically equal to 0 and this implies that $f = 0$.

□

Corollary 10 (Cor. IV.23 in [1]). $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega)$*, for all* $1 \leq p < +\infty$*.*

8.1.5 Partition of unity

We conclude the list of preliminary results with a partition of unity theorem. We need, before, a property that we let as an exercise.

Exercise 2. *Let K be a compact set in $\mathbb{R}^n$. Let Ω_1 and Ω_2 be two open sets in $\mathbb{R}^n$, with $K \subseteq \Omega_1 \cup \Omega_2$ and $K_j \cap \Omega_j \neq \emptyset$, for $j = 1, 2$. Show that there exists two compact sets $K_1 \subseteq \Omega_1$ and $K_2 \subseteq \Omega_2$ such that $K = K_1 \cup K_2$.*

Hint. First of all, if $\Omega_1 \cap \Omega_2 = \emptyset$ then it is sufficient to take $K_j = K \cap \Omega_j$, for $j = 1, 2$. If $\Omega_1 \cap \Omega_2 \neq \emptyset$, for every $x \in K$, consider an open ball $B(x, r_x)$ such that,

$$\text{if } x \in K \setminus \Omega_j, \text{ then } B(x, 2r_x) \subseteq \Omega_j, \text{ for } j = 1, 2,$$

$$\text{if } x \in K \cap \Omega_1 \cap \Omega_2, \text{ then } B(x, 2r_x) \subseteq \Omega_1 \cap \Omega_2.$$

$\{B(x, r_x) \mid x \in K\}$ is an open covering of K. Take a finite subcovering

$$B_1(x_1, r_1), \ldots, B_1(x_N, r_N)$$

Define

$$K_1 = K \cap \Big(\bigcup_{x_i \in \Omega_1} \overline{B_1(x_i, r_i)} \Big) \qquad \text{and} \qquad K_2 = K \cap \Big(\bigcup_{x_i \in \Omega_2} \overline{B_1(x_i, r_i)} \Big).$$

Theorem 32 (Th. 1.2.3 in [3])**.** *Let K be a compact set in $\mathbb{R}^n$. Let $\Omega_1, \ldots, \Omega_N$ be open sets in $\mathbb{R}^n$, with $K \subseteq \bigcup_{j=1}^N \Omega_j$.*

Then there exist $\varphi_1, \ldots, \varphi_N$ with, for all j, $\varphi_j \in C_0^\infty(\Omega_j)$ such that,

$$\sum_{j=1}^N \varphi_j(x) = 1, \qquad \textit{for all} \quad x \in K.$$

Proof. Using the exercise we can find $K_1, \ldots, K_N$ compact sets, with, for all j, $K_j \subseteq \Omega_j$ and $\cup_j K_j = K$. We consider, for all j, $\psi_j \in C_0^\infty(\Omega_j)$, such that $\psi_j = 1$ in a neighborhood of K_j. We set

$$\begin{aligned}
&\varphi_1 = \psi_1, \\
&\varphi_2 = \psi_2(1 - \psi_1), \\
&\varphi_3 = \psi_3(1 - \psi_2)(1 - \psi_1), \\
&\vdots \\
&\varphi_N = \psi_N(1 - \psi_{N-1})(1 - \psi_{N-2}) \cdot \ldots \cdot (1 - \psi_1).
\end{aligned}$$

By induction, it is possible to prove that

$$\varphi_1 + \varphi_2 + \ldots + \varphi_N = 1 - (1 - \psi_1) \cdot \ldots \cdot (1 - \psi_N),$$

and the conclusion follows. ☐

References

[1] Brezis, Haïm. "Analyse fonctionnelle. (French) [Functional analysis] Théorie et applications. [Theory and applications]". Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree] Masson, Paris, 1983. xiv+234 pp.

[2] Hewitt, Edwin; Stromberg, Karl. "Real and abstract analysis. A modern treatment of the theory of functions of a real variable". Third printing. Graduate Texts in Mathematics, No. 25. Springer-Verlag, New York-Heidelberg, 1975. x+476 pp.

[3] Hörmander, Lars. "Linear partial differential operators". Die Grundlehren der mathematischen Wissenschaften, Bd. 116 Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg 1963 vii+287 pp.

[4] Katsuura, Hidefumi. *Continuous nowhere-differentiable functions–an application of contraction mappings.* Amer. Math. Monthly 98, no. 5 (1991), 411–416.

[5] Kolmogorov, A. N.; Fomin, S. V. "Introductory real analysis". Revised English edition. Translated from the Russian and edited by Richard A. Silverman Prentice-Hall, Inc., Englewood Cliffs, N.J. 1970 xii+403 pp. *Italian edition: "Elementi di teoria delle funzioni e di analisi funzionale". Editori Riuniti Univ. Press, 2012.*

[6] McCarthy, John. *An everywhere continuous nowhere differentiable function.* Amer. Math. Monthly 60 (1953), 709. *Free download at the address* http://jmc.stanford.edu/articles/weierstrass/weierstrass.pdf

[7] Rudin, Walter. "Principles of mathematical analysis". Third edition. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976. x+342 pp. *Free download at the address* https://notendur.hi.is/vae11/%C3%9Eekking/principles_of_mathematical_analysis_walter_rudin.pdf

[8] Rudin, Walter. "Real and complex analysis". Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp.

[9] Stein, Elias. "Singular integrals and differentiability properties of functions". Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970 xiv+290 pp.