# HyPro: A C++ library of state set representations for hybrid systems reachability analysis

Stefan Schupp

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# Outline

# Hybrid systems

*"hybrid: [...] A thing made by combining two different elements."*
*Oxford dictionary*

Hybrid systems are systems combining discrete and continuous behavior.

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Hybrid systems are systems combining discrete and continuous behavior. They can be found in

- physical processes (bouncing ball, freezing water, ...)
- digital controllers for continuous systems (avionics, automotive, automated plants) $\rightarrow$ cyber-physical systems

As they interact and possibly modify the surrounding environment they are often safety critical.

# Hybrid systems reachability analysis

**Reachability problem (for hybrid systems)**

The reachability problem is the problem to decide whether a state is reachable in a hybrid system from a set of initial states.

Testing

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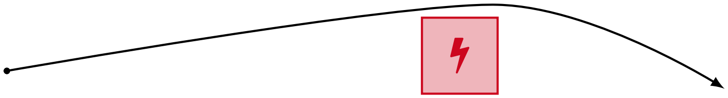

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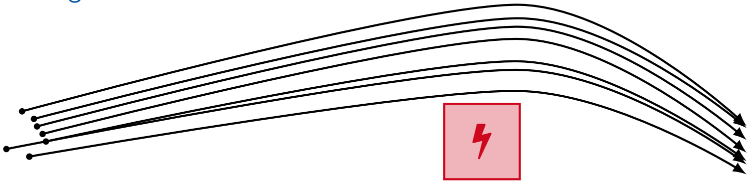

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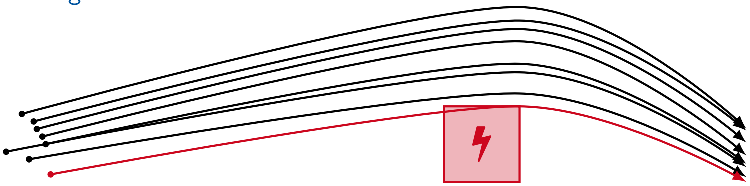

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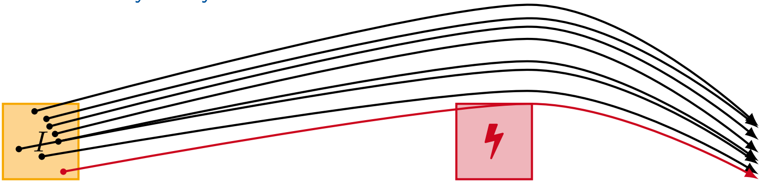

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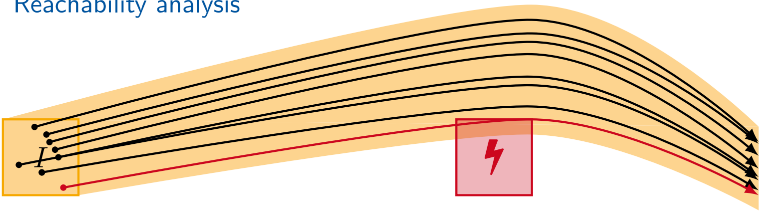

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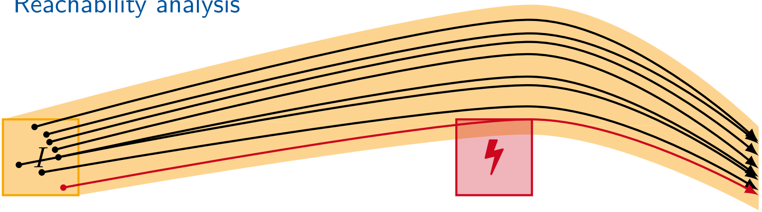


Problem: In general undecidable.

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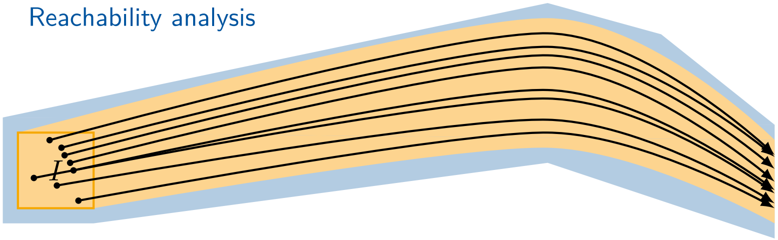


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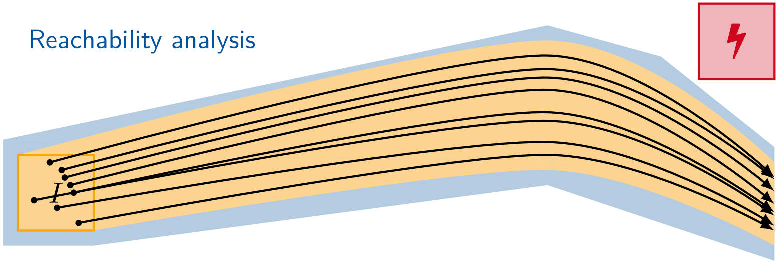


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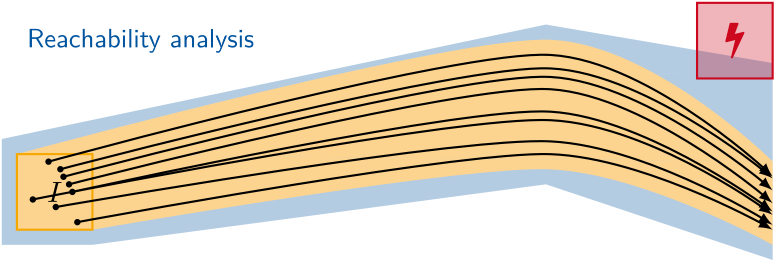


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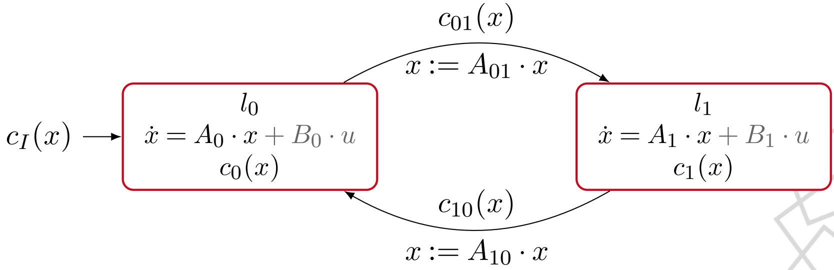
# Hybrid automata

Hybrid systems can be modeled by hybrid automata
Here: linear hybrid automata




A finite set of locations $Loc$

# Hybrid automata

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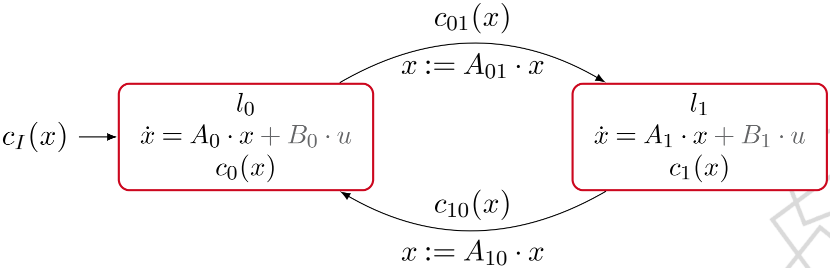


A vector of variables $x$

# Hybrid automata

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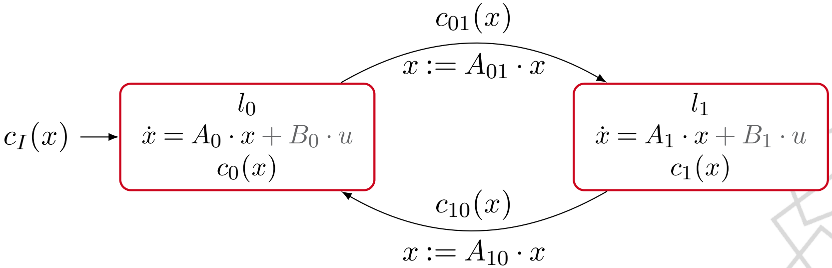


Flow: $Loc \rightarrow Pred_{Var \cup \dot{Var}}$

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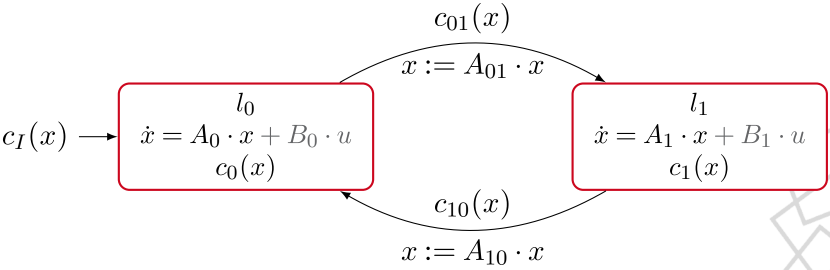


Invariant: $Loc \rightarrow Pred_{Var}$

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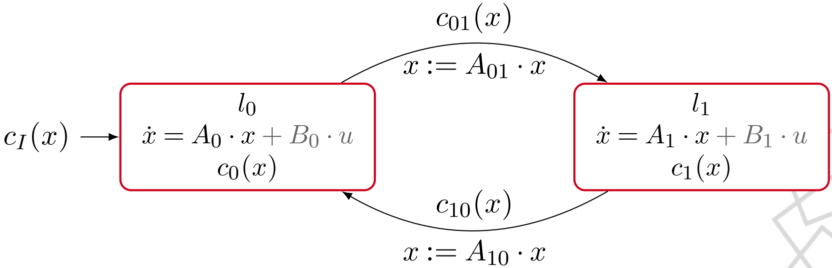


Transitions: $Edge \subseteq Loc \times Pred_{Var} \times Pred_{Var \cup Var'} \times Loc$

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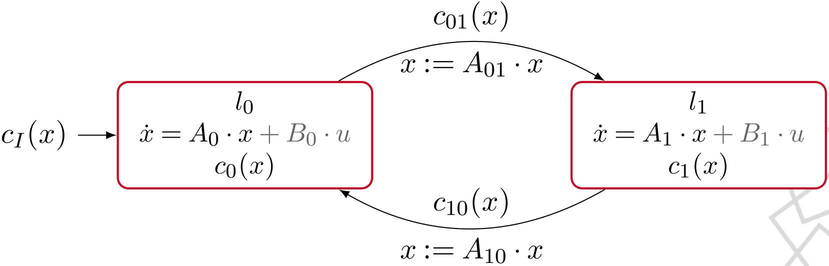
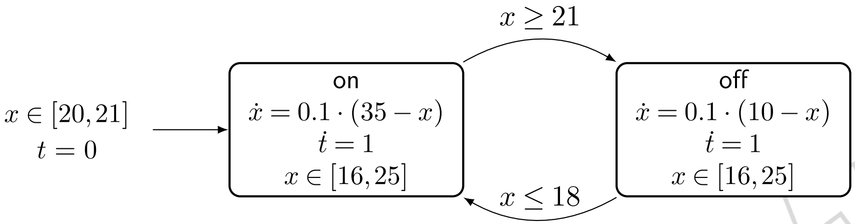



An initial set $Loc \to Pred_{Var}$

# Hybrid automata – example

Simplified model of a thermostat[1]:




[1] https://www.digitalcity.wien/even-thermostats-have-a-heart/

# Reachability analysis algorithm

## Basic iterative reachability analysis approach

**Input:** Set Init of initial states.
**Output:** Set R of reachable states.

**Algorithm:**

$$
\begin{array}{l}
R^{\text{new}} := \text{Init}; \\
R := \emptyset; \\
\text{while } (R^{\text{new}} \neq \emptyset)\{ \\
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Question: How to compute Reach for (linear) hybrid systems?

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Question: How to compute Reach for (linear) hybrid systems?
Answer: Alternatingly compute time- and jump-successor states.

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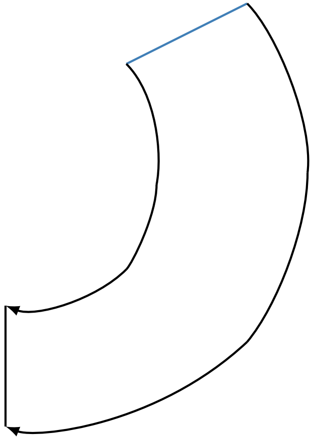

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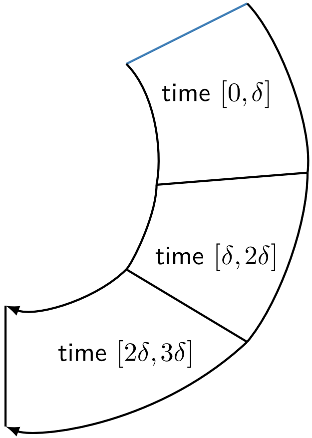

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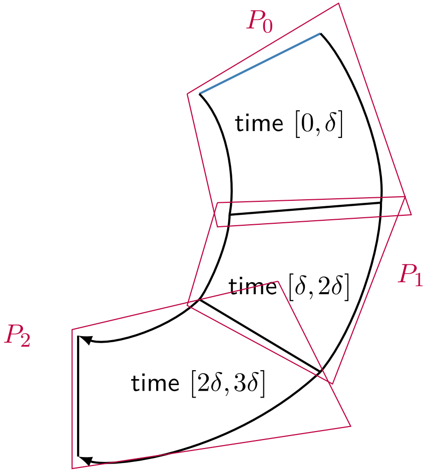

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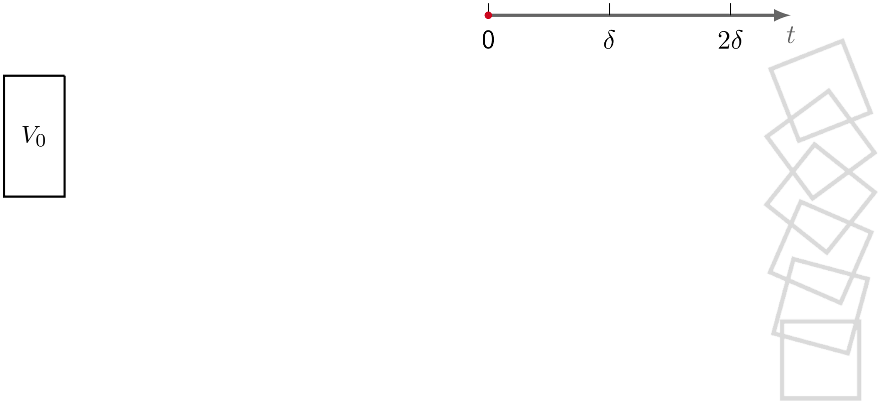

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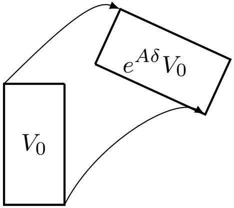


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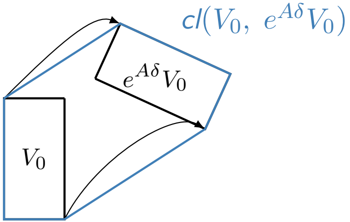


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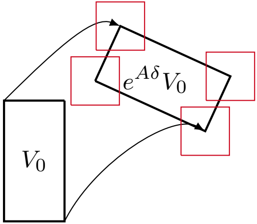


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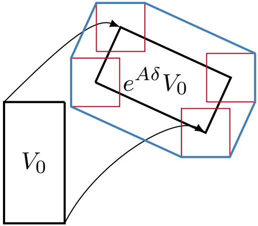


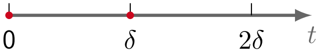

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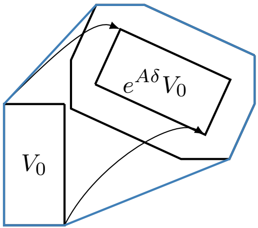
$P_0 = cl(V_0,\ e^{A\delta}V_0 \oplus \mathcal{V}_A)$

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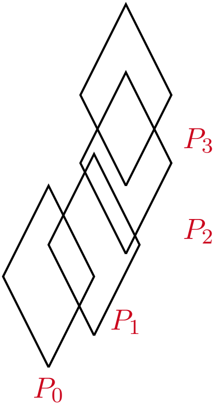

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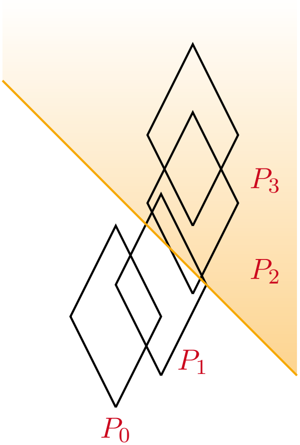

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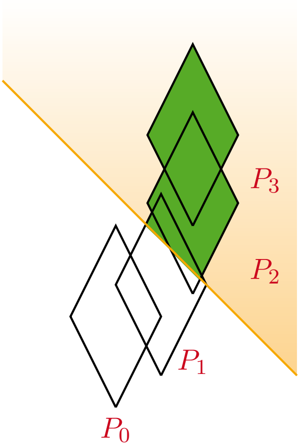

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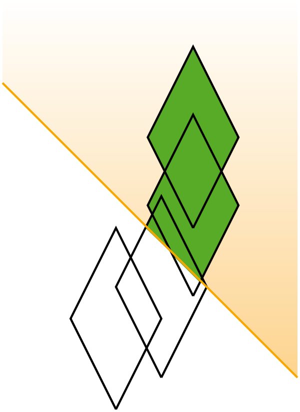

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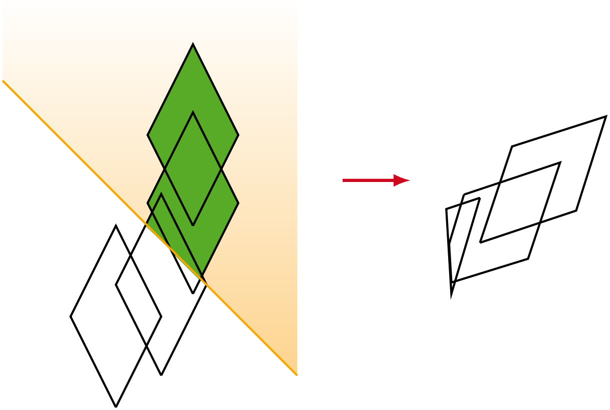

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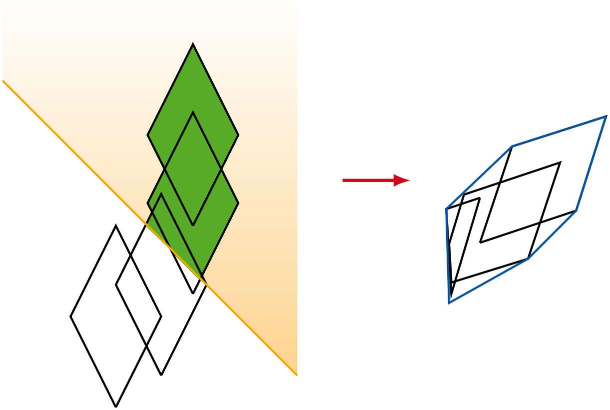
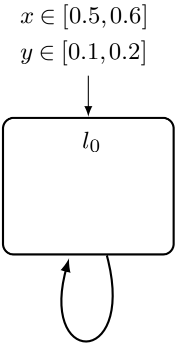
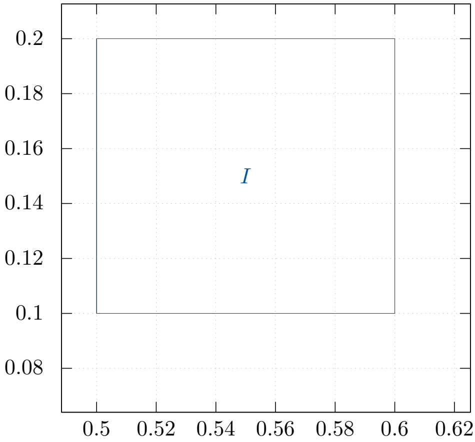


# Example - linear hybrid automata



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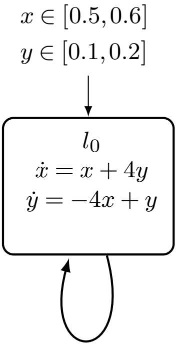


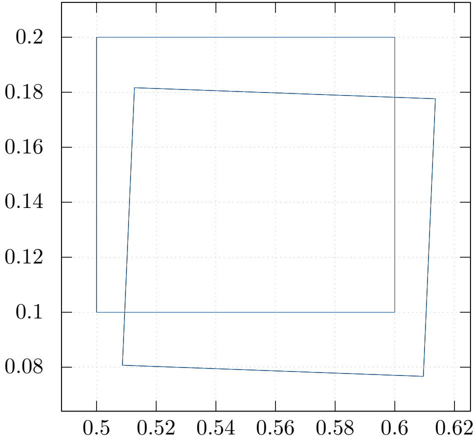


linear transformation: $e^{\delta A} \cdot I$

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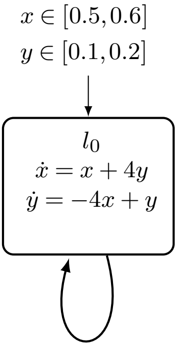
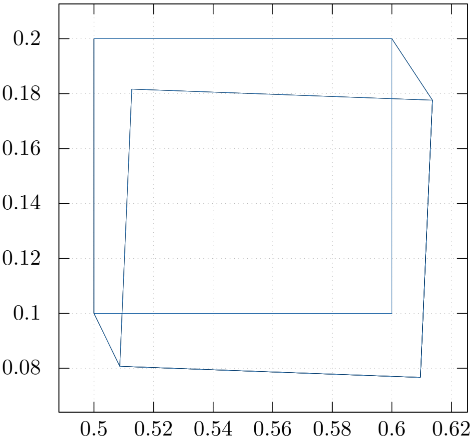
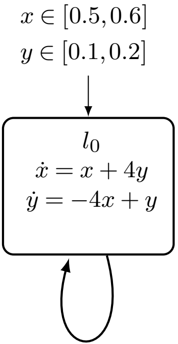
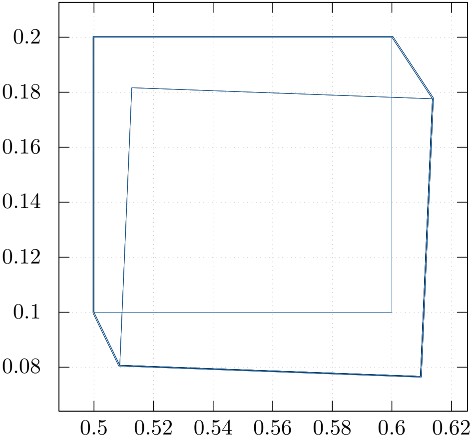



`union`

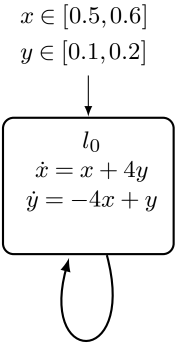
# Example - linear hybrid automata

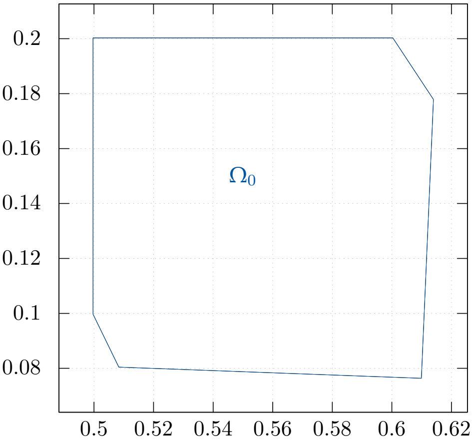



Minkowski sum

# Example - linear hybrid automata



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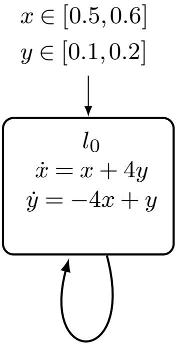


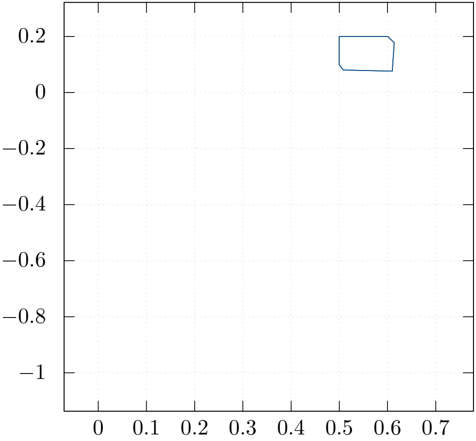

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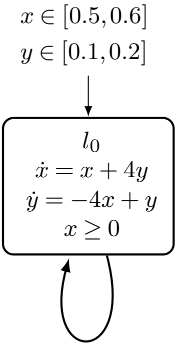
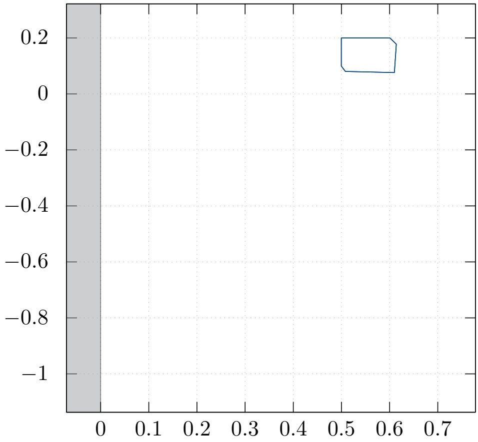
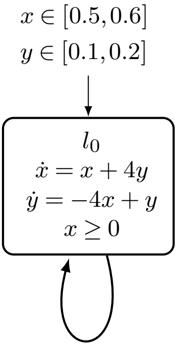
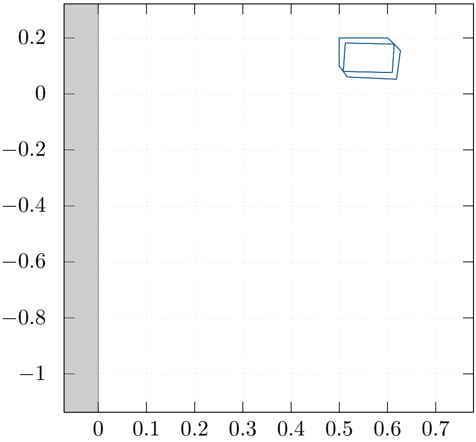



`intersection:` $Inv(l_0) \cap \Omega_i$

# Example - linear hybrid automata




`linear transformation:` $\Omega_{i+1} = e^{\delta A} \cdot \Omega_i$

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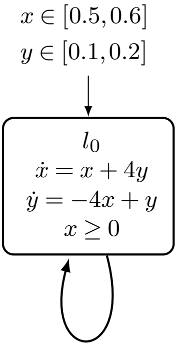


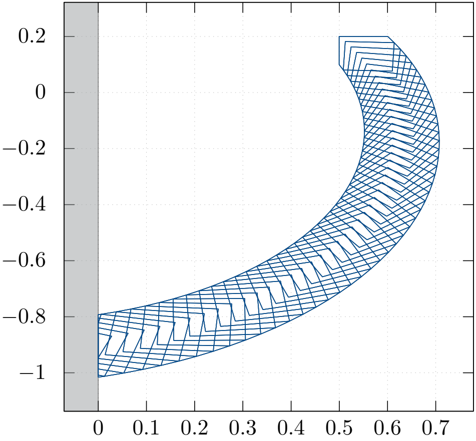


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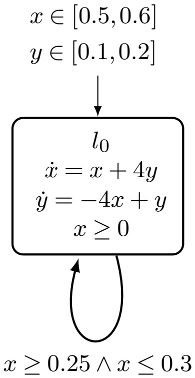


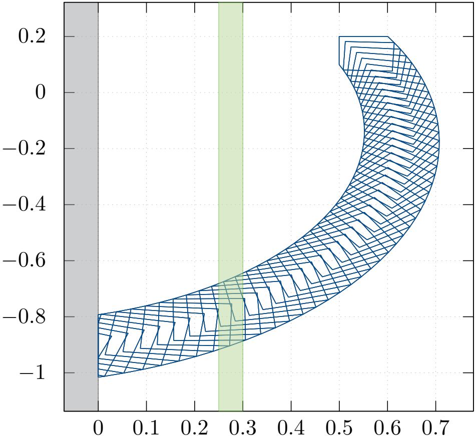


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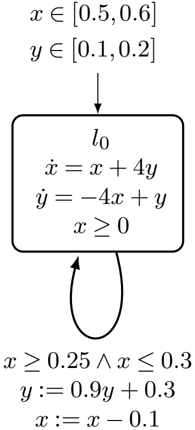
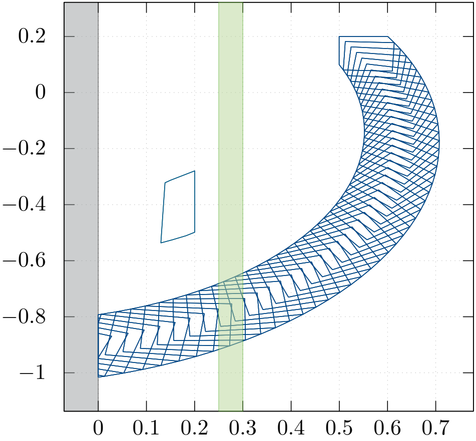
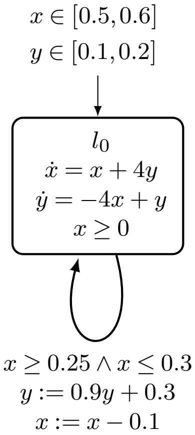
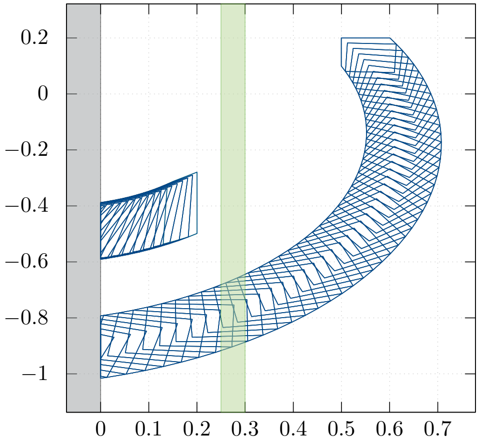



`linear transformation:` $I' := reset(\Omega_i)$

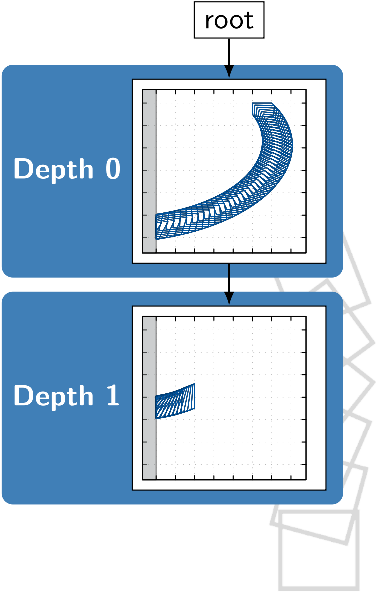
# Example - linear hybrid automata




`linear transformation:` $\Omega_{i+1} = e^{\delta A} \cdot \Omega_i$

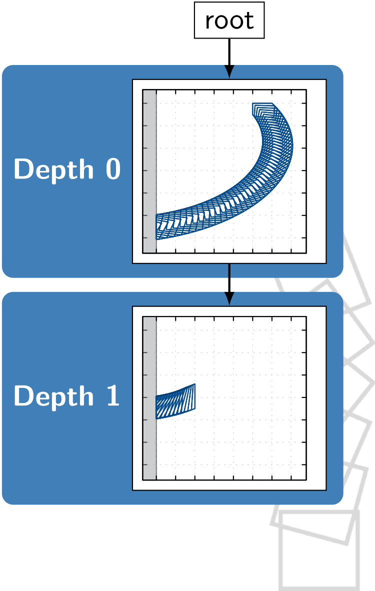
# Induced search tree



# Induced search tree

The induced search tree depends on:

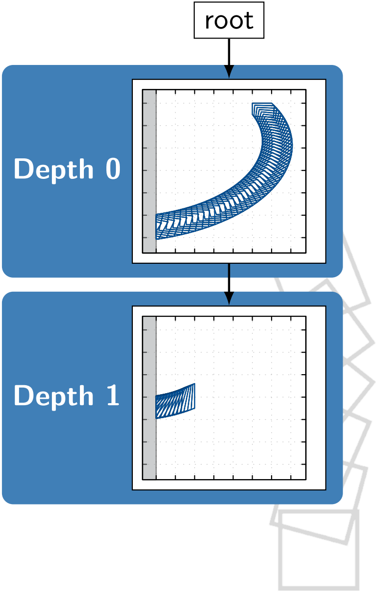
- The model itself
- Bounds (jump depth, time horizon)



# Induced search tree

The induced search tree depends on:

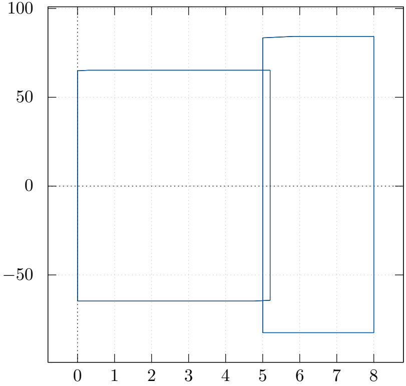
- The model itself
- Bounds (jump depth, time horizon)
- Time step size
- State set representation
- Aggregation settings



# Analysis parameters – examples

The *precision* and *running time* depends on several parameters, e.g.,

- Time step size $\delta$




$\delta = 5$

# Analysis parameters – examples

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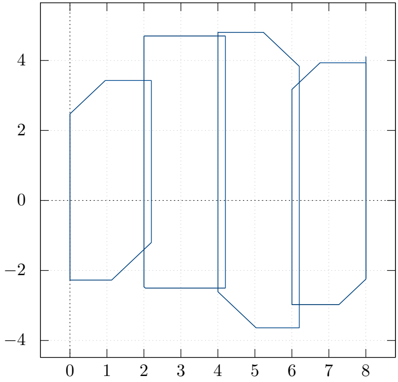


$\delta = 2$

# Analysis parameters – examples

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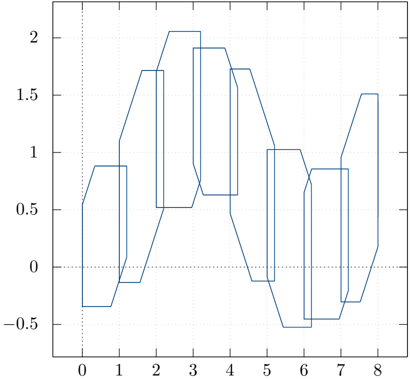


$\delta = 1$

# Analysis parameters – examples

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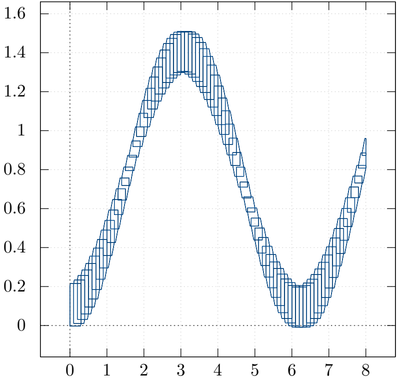


$\delta = 0.1$

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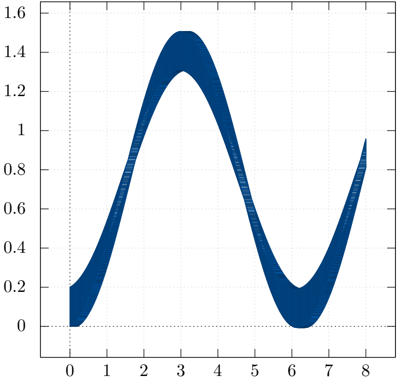


$\delta = 0.01$

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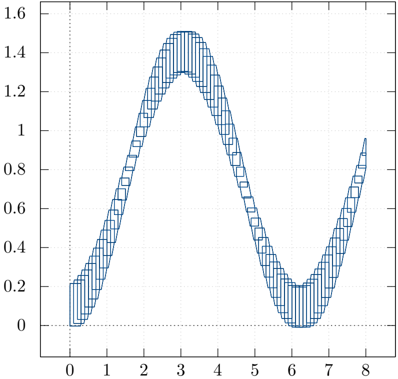


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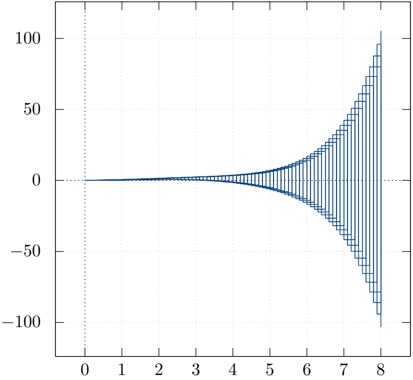


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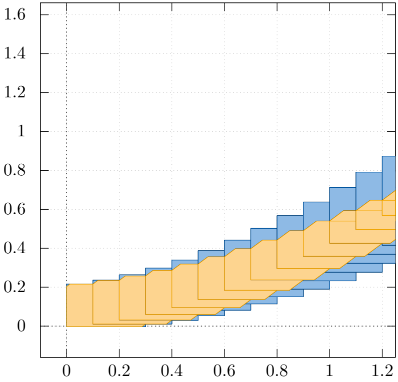


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The *precision* and *running time* depends on several parameters, e.g.,

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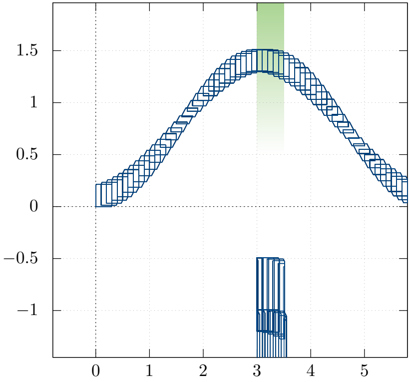
  - Default behavior
    - + No additional effort
    - − No control of number of discrete successors

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    - + Only one discrete successor
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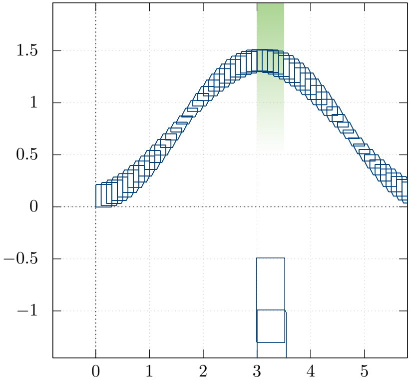


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# Sets & required set operations

Required: State set representation.

Problem: There are several ways to represent sets (see next slides).

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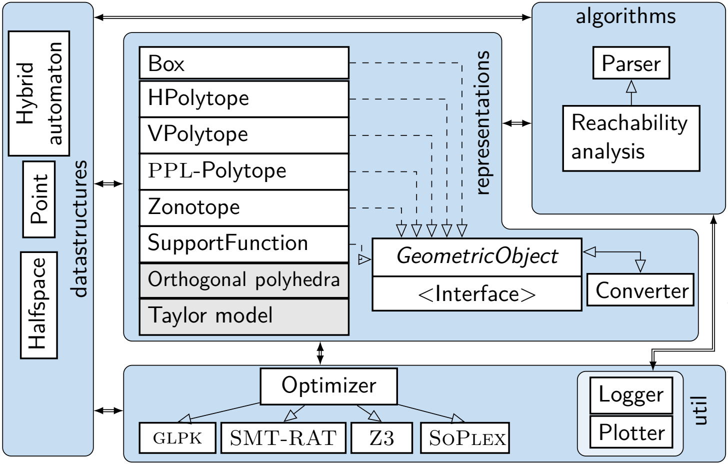
Goal: Unify available state set representations with a common interface.

# HyPro[2]




[2] [SÁBMK17]

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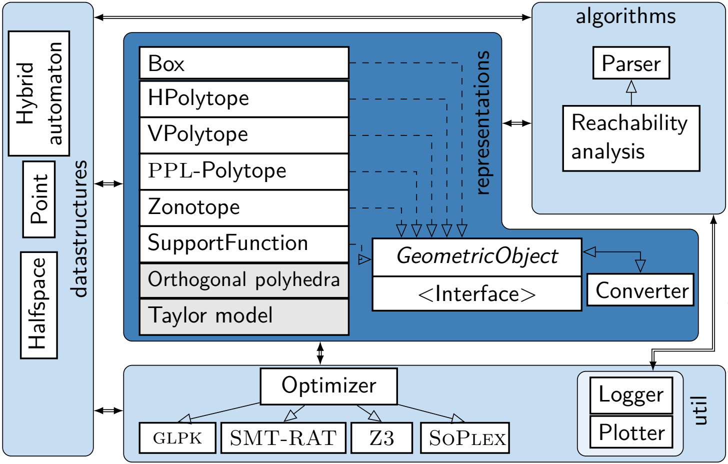
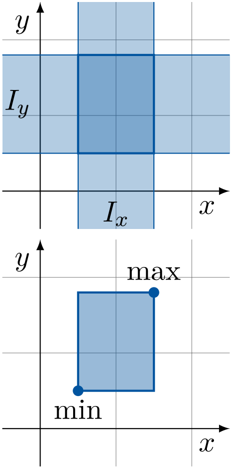



[2][SÁBMK17]

# Implemented state set representations

- boxes [MKC09]



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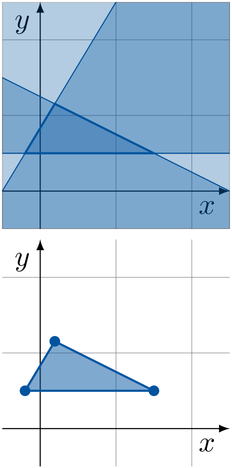

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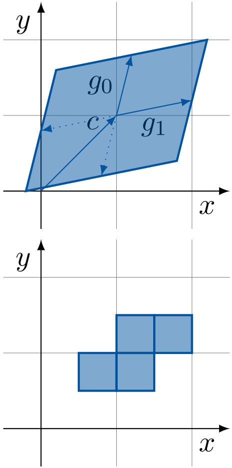

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- boxes [MKC09]
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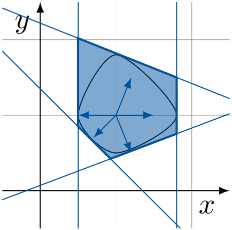
- support functions [LGG10]
- Taylor models [CÁS12]



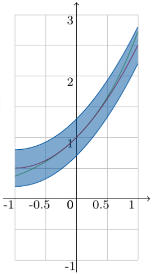




Image: Xin Chen

# `GeometricObjectBase` interface

Set operations:

| | |
|---|---|
| `X.affineTransformation(matrix A, vector b)` | $AX + b$ |
| `X.minkowskiSum(geometricObject Y)` | $X \oplus Y$ |
| `X.intersectHalfspaces(matrix A, vector b)` | $X \cap \{y \mid Ay \leq b\}$ |
| `X.satisfiesHalfspaces(matrix A, vector b)` | $X \cap \{y \mid Ay \leq b\} \neq \emptyset$ |
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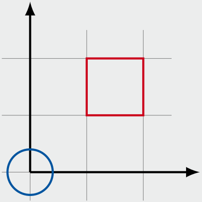
## Recap: Minkowski sum (dilation)

$A \oplus B = \{x \mid x = a+b, a \in A, b \in B\}$



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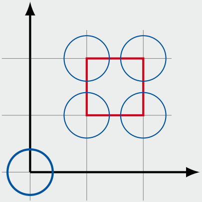

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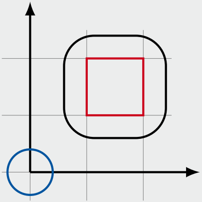

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Set utility functions:

```
dimension()
empty()
vertices()
project(vector<dimensions> d)
contains(point p)
```

conversion operations

reduction functions

# Operations – complexity

Computational effort required for the most commonly used operations for different representations:

| | $\cdot \bigcup \cdot$ | $\cdot \bigcap \cdot$ | $\cdot \oplus \cdot$ | $A(\cdot)$ |
|---|---|---|---|---|
| Box | | | $+$ | |
| $\mathcal{H}$-polytope | - | $+$ | - | - |
| $\mathcal{V}$-polytope | $+$ | - | $+$ | $+$ |
| Zonotope | | | $+$ | $+$ |
| Support function | $+$ | - | $+$ | $+$ |

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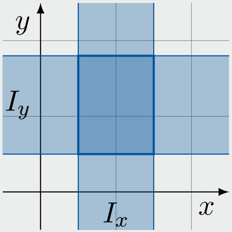
# Boxes

Boxes are one of the simplest ways to represent a set:

**Definition: box [MKC09]**

A box $\mathcal{B}$ of dimension $n$ is defined as an ordered vector of intervals




$$\mathcal{B} = (I_0, \ldots, I_n), I_i \in \mathbb{I}$$

Where $\mathbb{I}$ is the set of all real-valued intervals

$$I_i = \{x \mid l \leq x \leq u\} \; l, u \in \mathbb{R},$$

we write $I_i = [l, u] \in \mathbb{I}$

# Boxes – operations

Intersection:

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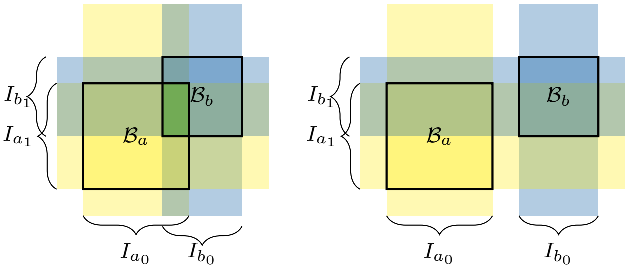
For boxes:

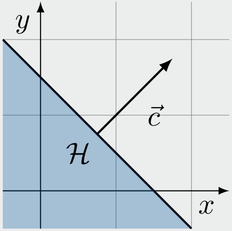
$$\mathcal{B}_c = I_{a_0} \cap I_{b_0}, \ldots, I_{a_n} \cap I_{b_n}$$



# Boxes – operations

Intersection with a half-space (e.g. guards, invariants):

## Recap: half-space




A half-space $\mathcal{H} \in \mathbb{R}^n$ contains all points

$$\mathcal{H} = \{x \mid \vec{c}^T \cdot x \leq d,\ \vec{c} \in \mathbb{R}^n,\ d \in \mathbb{R}\}$$

Example:

$$\mathcal{H} = \left\{ x \,\middle|\, \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \cdot x \leq 1.5 \right\}$$

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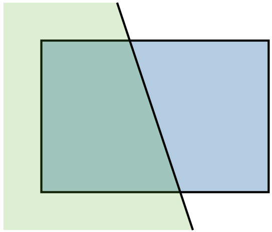

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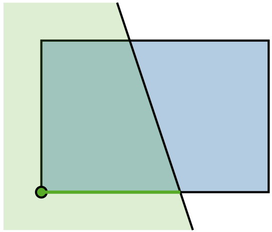

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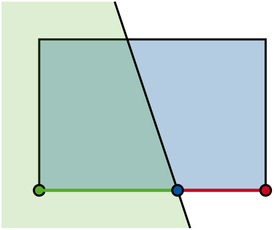

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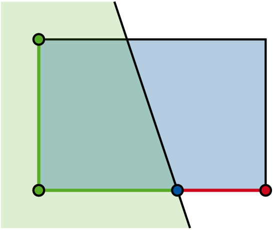

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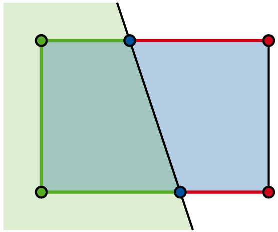

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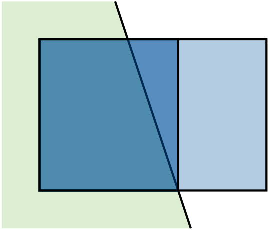

# Excursion: Interval Arithmetic[3]

Binary operations (general case):

$$X \odot Y = \{x \odot y \mid x \in X, y \in Y\}, X, Y \in \mathbb{I}$$

**Example (Basic arithmetic operations)**

Addition: $[4,5] \quad + \quad [-1,2]$

[3] See e.g., [MKC09] for details.

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[3]See e.g., [MKC09] for details.

# Excursion: Interval Arithmetic[3]

Binary operations (general case):

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**Example (Basic arithmetic operations)**

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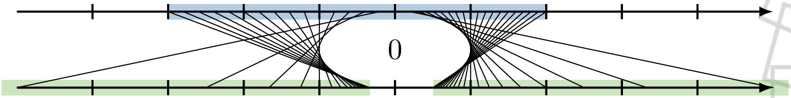


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If one interval becomes empty, the constraint is not satisfiable.

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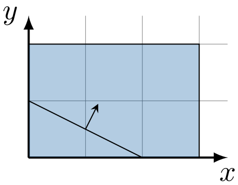
Assume $\mathcal{B} = [0,3] \times [0,2]$ and a constraint $c\colon x + 2 \cdot y \leq 2$.

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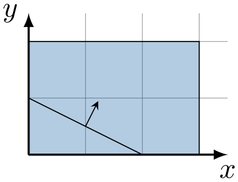


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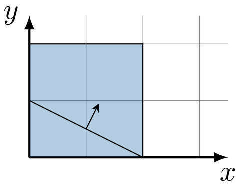


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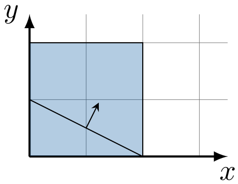


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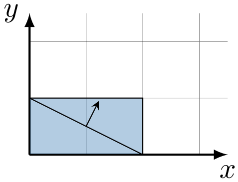


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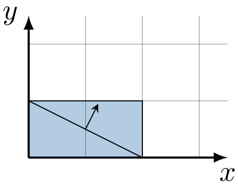


Note: termination not guaranteed due to new intervals.

But: For single linear constraints, a single iteration suffices[4].

[4] See [Sch19] for a proof.

# Boxes – operations

Union:

$$\mathcal{B}_c = \mathcal{B}_a \cup \mathcal{B}_b = \{x \mid x \in \mathcal{B}_a \vee x \in \mathcal{B}_b\}$$

Note: The union of two convex sets is not necessarily convex $\rightarrow$ we use the closure ($cl$) of the union.

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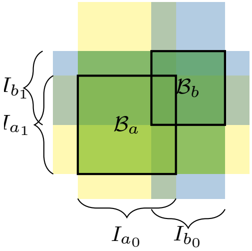
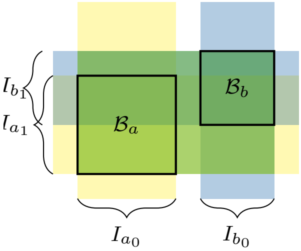
$$\begin{aligned}\mathcal{B}_c &= cl(I_{a_0} \cup I_{b_0}), \ldots, cl(I_{a_n} \cup I_{b_n}) \\ &= [\min(I_{a_{0_l}}, I_{b_{0_l}}), \max(I_{a_{0_u}}, I_{b_{0_u}})], \ldots, [\min(I_{a_{n_l}}, I_{b_{n_l}}), \max(I_{a_{n_u}}, I_{b_{n_u}})]\end{aligned}$$



# Boxes – operations

Minkowski-sum:

$$\mathcal{B}_c = \mathcal{B}_a \oplus \mathcal{B}_b = \{x \mid x = x_a + x_b, x_a \in \mathcal{B}_a, x_b \in \mathcal{B}_b\}$$

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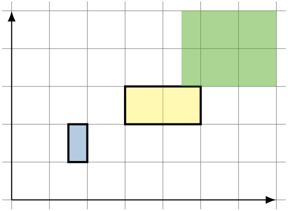

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Linear transformation:

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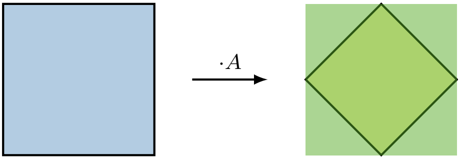
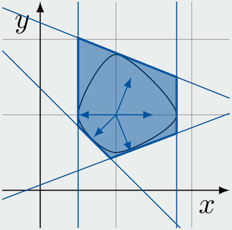
Approaches:

- Naive (conversion): apply $A$ on all vertices, re-convert to box
- Utilize interval arithmetic



# Support functions

## Definition: support function




The support function $\rho_\Omega$ of a n-dimensional set $\Omega \in \mathbb{R}^n$ is defined as

$$\rho_\Omega : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$$
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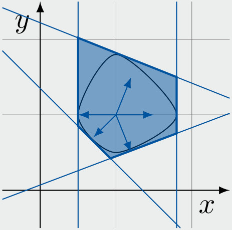


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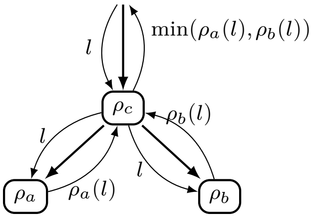
Properties:

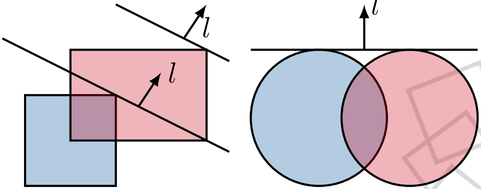
- implemented as tree structure (see next slides)
- operations are cheap, reduced overhead
- scale well in higher dimensions
- well developed (see e.g. [LGG10, FKL13, FGD$^+$11, LG09])

# Support functions – operations [LGG10]

Most commonly used operations during reachability analysis:

- Intersection: $\rho_c(l) = \min(\rho_a(l), \rho_b(l))$



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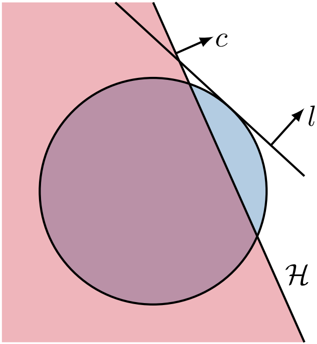
Most commonly used operations during reachability analysis:

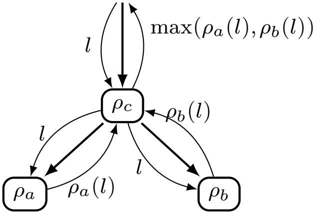
- Intersection with a half-space $\mathcal{H} = c^T \cdot x \leq d$ (e.g. guards, invariants): $\rho_c(l) = \min(\rho_a(l), \mathcal{H}(l))$,
  where $\mathcal{H}(l) = \begin{cases} d & \text{when } l = c \\ \infty & \text{else} \end{cases}$

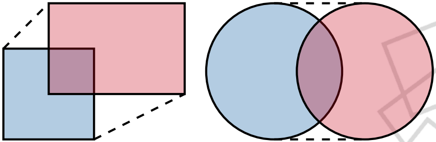


# Support functions – operations [LGG10]

Most commonly used operations during reachability analysis:

- Union: $\rho_c(l) = \max(\rho_a(l), \rho_b(l))$



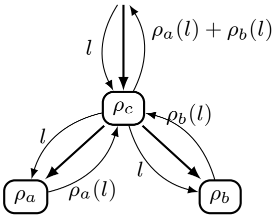



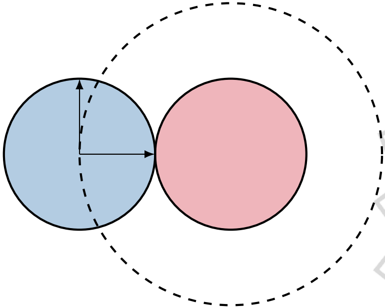
Note: The union operation on a set of support functions returns the supporting hyperplane of the convex hull of the set of underlying sets.

# Support functions – operations [LGG10]

Most commonly used operations during reachability analysis:

- Minkowski-sum: $\rho_c(l) = \rho_a(l) + \rho_b(l)$



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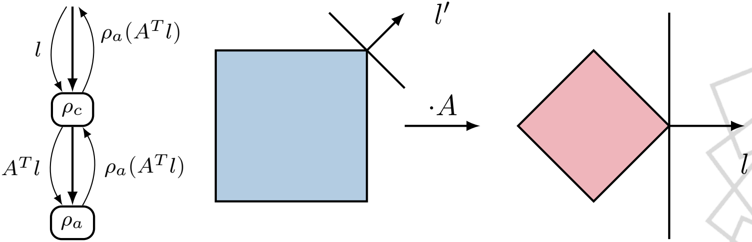

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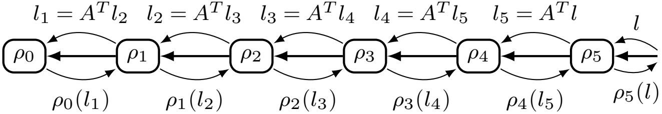
The tree structure in combination with our domain-specific knowledge allows for several optimizations:

- collect sequences of linear transformations



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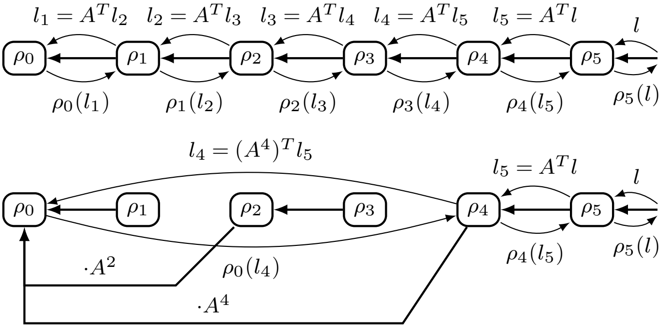

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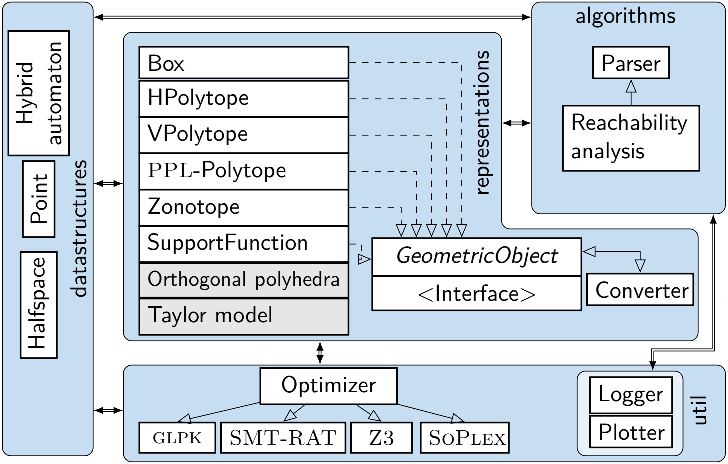
- remove intersections which have no effect
- reduce tree upon discrete jump (templated evaluation)

# HyPro[2]




[2][SÁBMK17]

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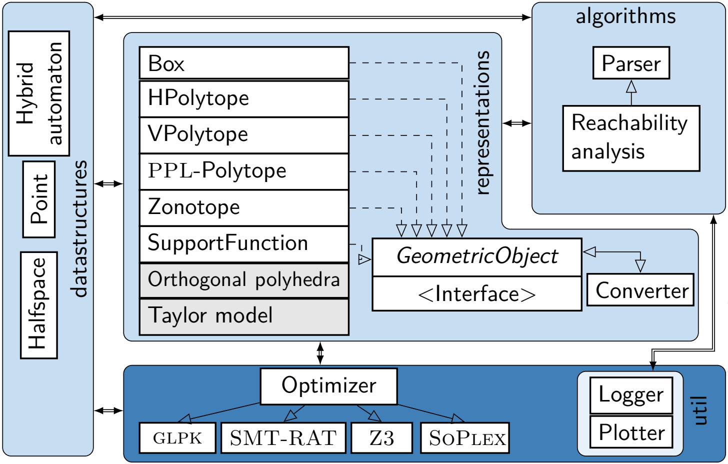



[2] [SÁBMK17]

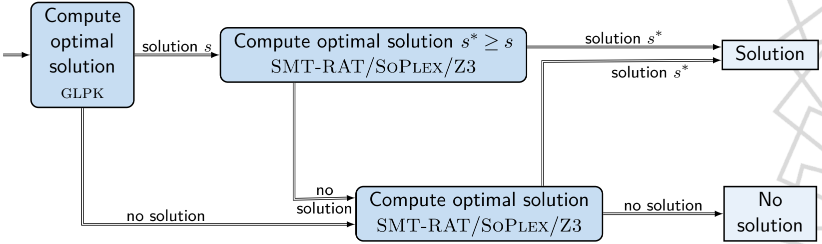
# Linear optimization

HyPro can use different number implementations via templates (supported: `cln::cl_RA`, `mpq_class`, `double`).

Obstacles:

- inexact linear optimization not suitable
- exact linear optimization expensive

⇝ combined application



# Utility

Additional features of HyPro:

- datastructures for e.g. hybrid automata, state, point, halfspace
- parser for Flow*-based syntax
- gnuplot plotting interface (pdf, eps and tex)
- logging

Reachability analysis methods:

- Linear hybrid automata
- Singular automata
- Rectangular automata
- Timed automata

Demo

# Thermostat[5]

We model and analyze a thermostat according to the following specifications:

- Can either be *on* (initially) or *off*
- Temperature $x$ changes accordingly: $\dot{x} = 50 - x$ (on), $\dot{x} = 10 - x$ (off)
- Switches from on to off when $x \in [20, 25]$
- Switches off to on when $x \in [16, 18]$

[5] https://www.digitalcity.wien/even-thermostats-have-a-heart/

# Outline

1 Introduction

2 HyPro
- State set representations

3 Short tutorial

4 Current research

# Applications

Extensions for reachability analysis based on HyPro:

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- CEGAR-based reachability analysis

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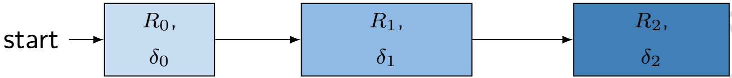

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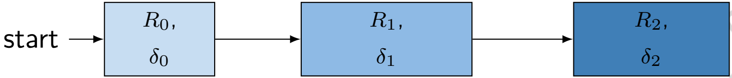
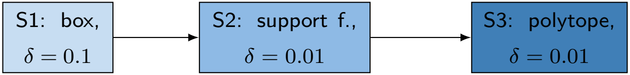



Depending on the application, order and choice of parameter settings matters!

# CEGAR-based reachability analysis - Example

Strategy:




Search tree:

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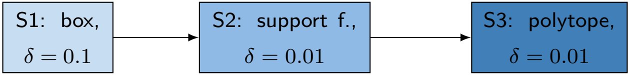


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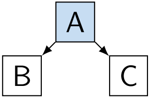

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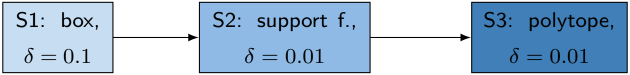


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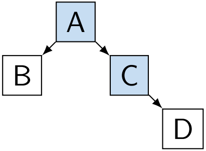

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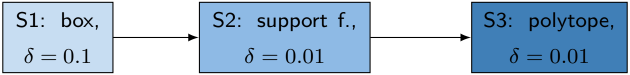


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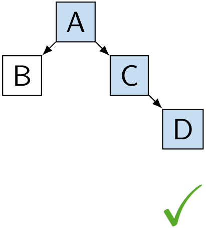

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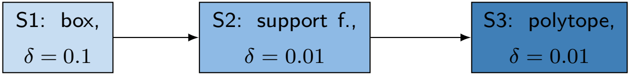


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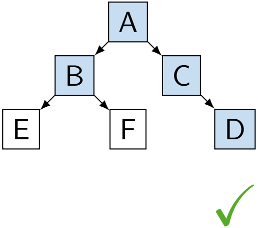

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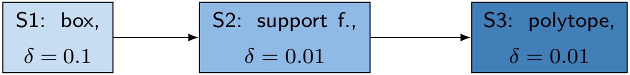


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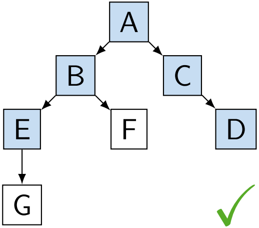

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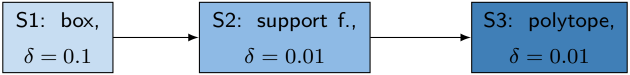


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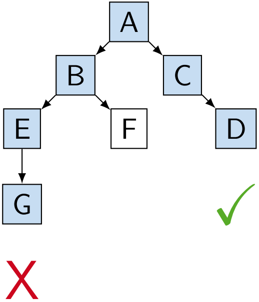

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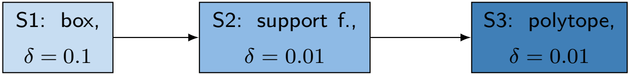


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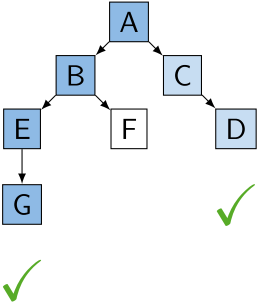

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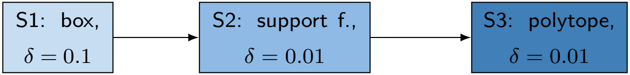


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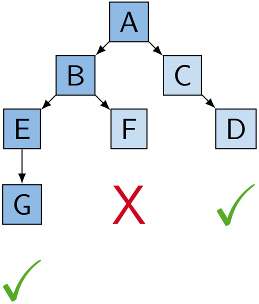

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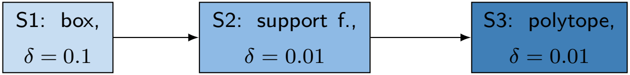


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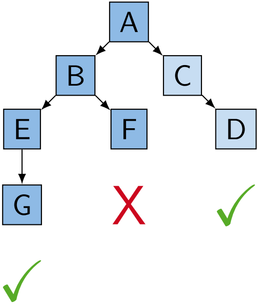

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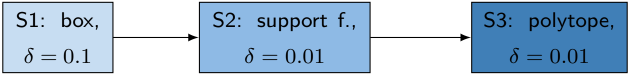


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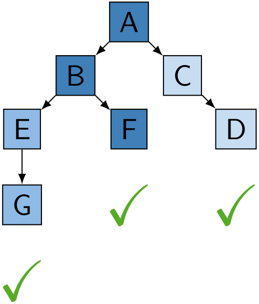

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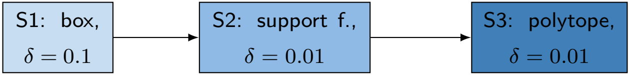
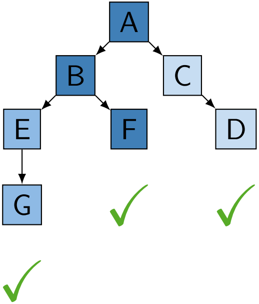



Search tree:




**Extension:** Parallelized search in different branches.

# Tree-updates

Variation of parameter settings influences the shape (number of child nodes) of the search tree.

- Aggregation settings
- Spurious branches (over-approximation)

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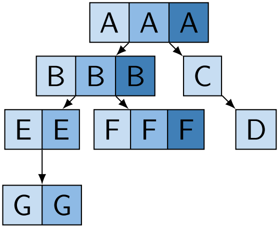
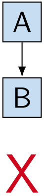
- Create multi-level tree



# Tree-updates

Update increases number of child nodes:



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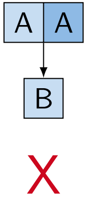

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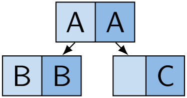

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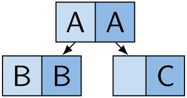
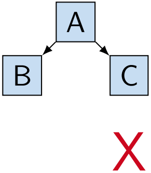



Update reduces number of child nodes:



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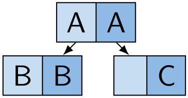


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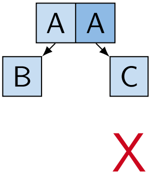

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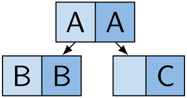


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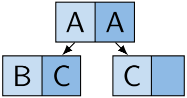
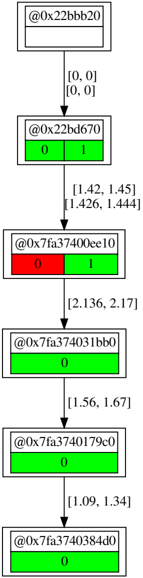


# Example: Bouncing ball



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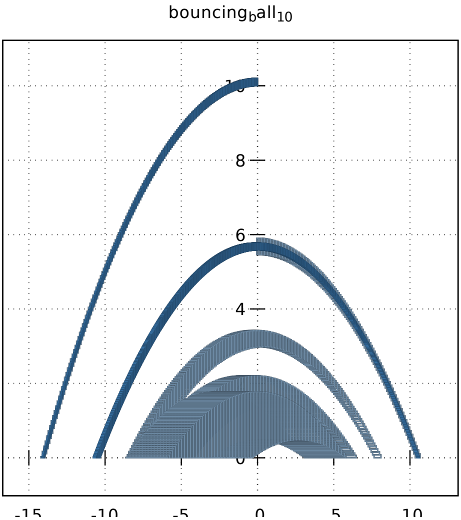

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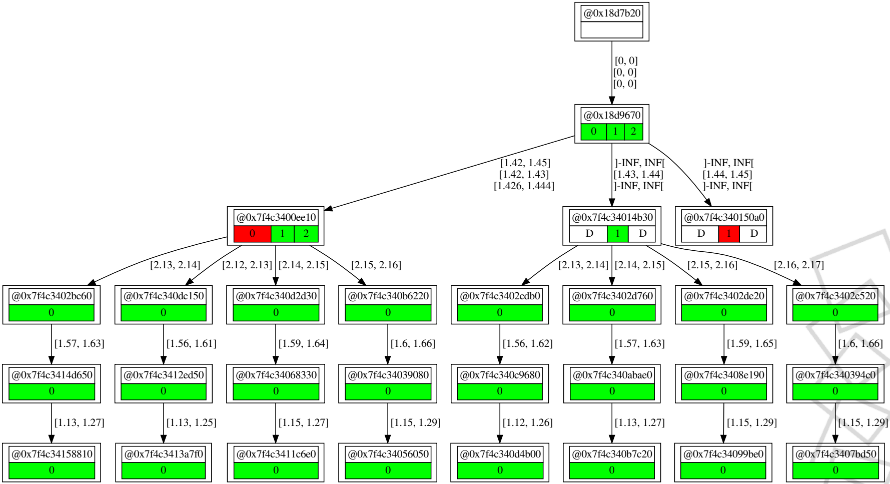

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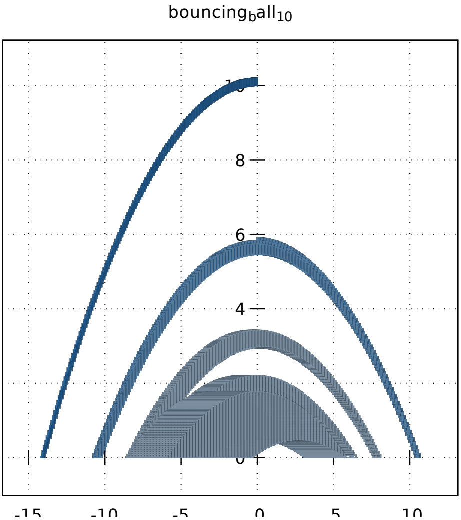

A free and open source library for hybrid systems reachability analysis

https://github.com/hypro/hypro

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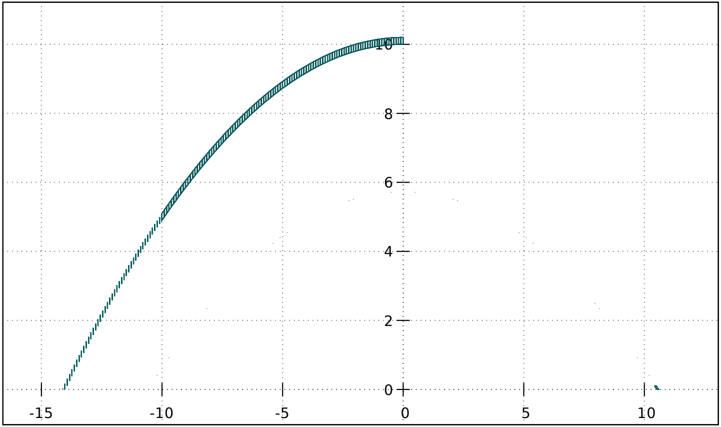
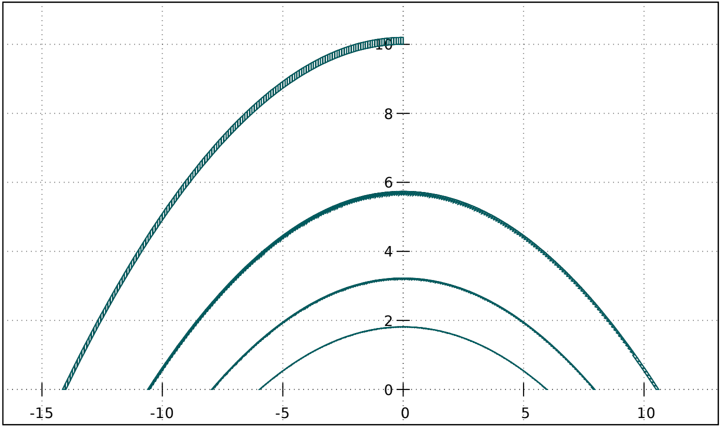
# Examples



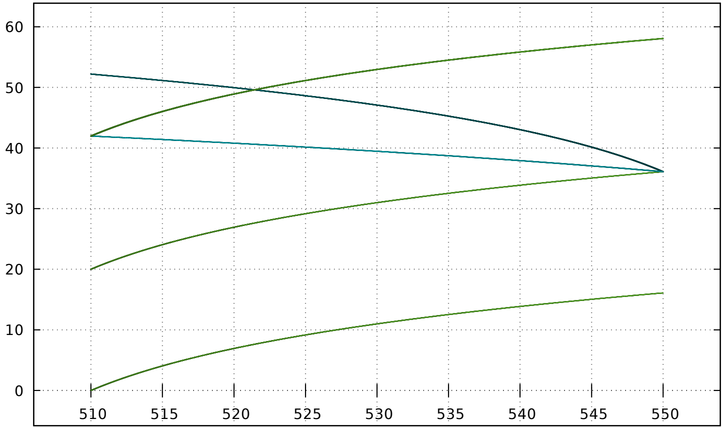

Bouncing ball, $\mathcal{V}$-polytopes with conversion to $\mathcal{H}$-polytopes for intersection, double glpk-only, $T = 3$, $\delta = 0.01$, 4 jumps

# Examples



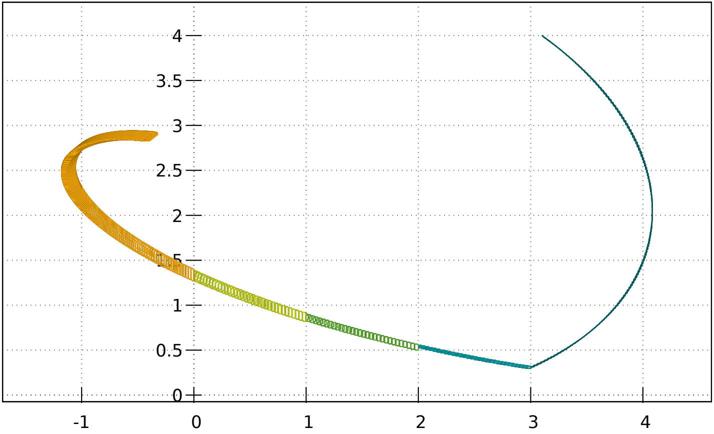

Bouncing ball, $\mathcal{V}$-polytopes with conversion to $\mathcal{H}$-polytopes for intersection, double glpk+SMT-RAT, $T = 3$, $\delta = 0.01$, 4 jumps

# Examples



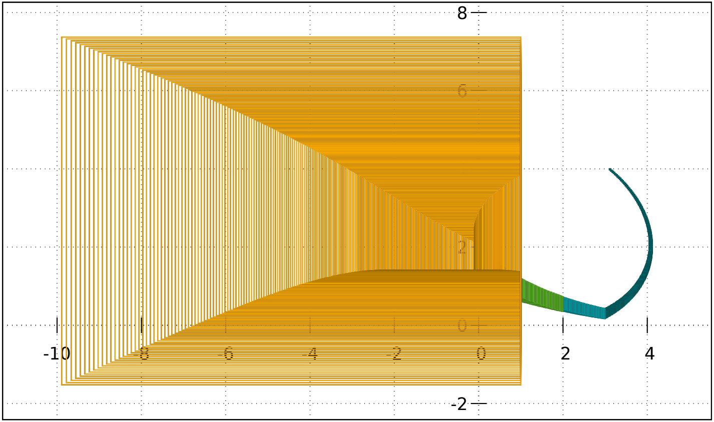

Rod reactor, box, double glpk-only, $T = 17$, $\delta = 0.01$, 2 jumps

# Examples



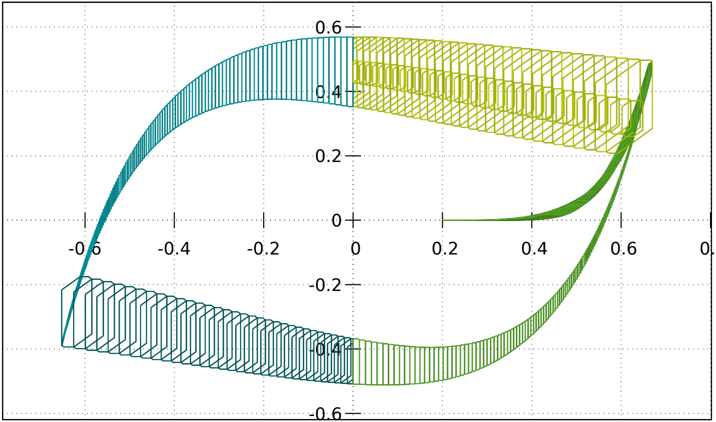

5-D switching system, support function, double glpk-only, $T = 0.2$, $\delta = 0.001$, 4 jumps

# Examples




5-D switching system, boxes, double glpk-only, $T = 0.2$, $\delta = 0.001$, 4 jumps

# Examples




Filtered oscillator, support function, double glpk-only, $T = 4$, $\delta = 0.01$, 5 jumps