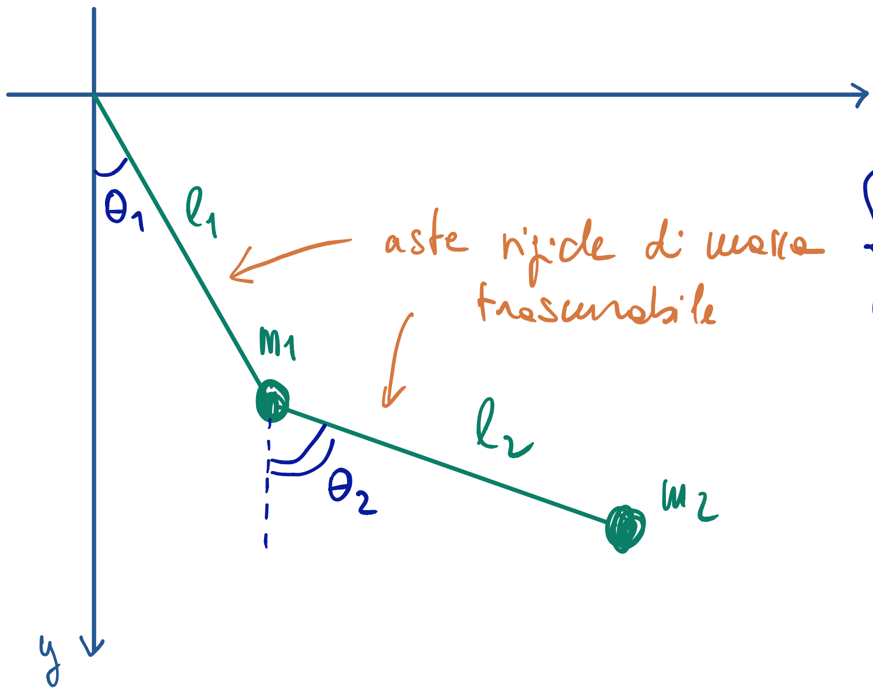
ESERCIZI

1 PENDOLO DOPPIO



$$\begin{cases} x_1 = l_1 \operatorname{sen}\theta_1 \\ y_1 = l_1 \cos\theta_1 \end{cases} \qquad \bar{r}_1(q_1,q_2) \quad \uparrow\theta_1 \quad \uparrow\theta_2$$

$$\begin{cases} x_2 = l_1 \operatorname{sen}\theta_1 + l_2 \operatorname{sen}\theta_2 \\ y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{cases} \qquad \bar{r}_2(q_1,q_2)$$

$\downarrow$

$n = 2$ gradi di libertà

1) Scrivere la Lagrangiana del sistema

$$L(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2) = T - V$$

$$T = \frac{1}{2} m_1 (\dot x_1^2 + \dot y_1^2) + \frac{1}{2} m_2 (\dot x_2^2 + \dot y_2^2)$$

$$\dot x_1 = l_1 \dot\theta_1 \cos\theta_1 \qquad \dot x_2 = l_1 \dot\theta_1 \cos\theta_1 + l_2 \dot\theta_2 \cos\theta_2$$

$$\dot y_1 = -l_1 \dot\theta_1 \operatorname{sen}\theta_1 \qquad \dot y_2 = -l_1 \dot\theta_1 \operatorname{sen}\theta_1 - l_2 \dot\theta_2 \operatorname{sen}\theta_2$$

$$T = \frac{1}{2} m_1 l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 \Big(l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \underbrace{(\cos\theta_1 \cos\theta_2 + \operatorname{sen}\theta_1 \operatorname{sen}\theta_2)}_{\cos(\theta_1-\theta_2)} \Big)$$

$$= \frac{1}{2}(m_1+m_2) l_1^2 \dot\theta_1^2 + \frac{1}{2} m_2 l_2^2 \dot\theta_2^2 + \frac{1}{2} m_2 \, 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1-\theta_2)$$

$$\frac{1}{2}\sum_{h,k=1}^{2} a_{hk}(\theta)\,\dot\theta_h\dot\theta_k \overset{\Updownarrow}{=} \frac{1}{2}\left(a_{11}\dot\theta_1^2 + a_{12}\dot\theta_1\dot\theta_2 + a_{21}\dot\theta_2\dot\theta_1 + a_{22}\dot\theta_2^2\right)$$

$$= \frac{1}{2}\left(a_{11}\dot\theta_1^2 + 2a_{12}\dot\theta_1\dot\theta_2 + a_{22}\dot\theta_2^2\right)$$

$\rightarrow$ Matrice cinetica
$$a(\theta_1,\theta_2) = \begin{pmatrix} (m_1+m_2)l_1^2 & m_2l_1l_2\cos(\theta_1-\theta_2) \\ m_2l_1l_2\cos(\theta_1-\theta_2) & m_2l_2^2 \end{pmatrix}$$

$$V = -m_1gy_1 - m_2gy_2 = -m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot\theta_1^2 + \frac{1}{2}m_2l_2\dot\theta_2^2 + m_2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 + (m_1+m_2)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2$$

2) Eq. di Lagrange

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_1} = \frac{d}{dt}\left((m_1+m_2)l_1^2\dot\theta_1 + m_2l_1l_2\cos(\theta_1-\theta_2)\dot\theta_2\right) =$$

$$= (m_1+m_2)l_1^2\ddot\theta_1 - m_2l_1l_2\sin(\theta_1-\theta_2)(\dot\theta_1-\dot\theta_2)\dot\theta_2 + m_2l_1l_2\cos(\theta_1-\theta_2)\ddot\theta_2$$

$$\frac{\partial L}{\partial\theta_1} = -m_2l_1l_2\sin(\theta_1-\theta_2)\dot\theta_1\dot\theta_2 - (m_1+m_2)gl_1\sin\theta_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_1} - \frac{\partial L}{\partial\theta_1} = (m_1+m_2)l_1^2\ddot\theta_1 + m_2l_1l_2\cos(\theta_1-\theta_2)\ddot\theta_2 + m_2l_1l_2\sin(\theta_1-\theta_2)\dot\theta_2^2 + (m_1+m_2)gl_1\sin\theta_1 = 0$$

eq. diff. non-lineari

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_2} - \frac{\partial L}{\partial\theta_2} = \dots$$

3) Determinare i pti di equil. del sistema, discutendo la loro stabilità.

In pto caso meccanico con forze conservative i pti di equil sono i pti stazionari dell'en. potenziale

$$V = -(m_1+m_2)gl_1\cos\theta_1 - m_2 g l_2 \cos\theta_2$$

$$\frac{\partial V}{\partial \theta_1} = (m_1+m_2)gl_1 \operatorname{sen}\theta_1 \quad \leftarrow \quad \theta_1 = 0, \pi$$

$$\frac{\partial V}{\partial \theta_2} = m g l_2 \operatorname{sen}\theta_2 \quad \leftarrow \quad \theta_2 = 0, \pi$$

⇝ Pti di equil. sono

$$(\theta_1^*, \theta_2^*) = (0, 0), \ (0, \pi), \ (\pi, 0), \ (\pi, \pi)$$

Pto stabile $\Longleftrightarrow$ è MIN (stretto) dell'en. POTENZIALE

$\Longleftrightarrow$ $\partial^2 V$ è (stretta) def. positiva

$$\frac{\partial^2 V}{\partial \theta_1^2} = (m_1+m_2)gl_1\cos\theta_1$$

$$\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} = 0 \quad \rightarrow \quad \partial^2 V = \begin{pmatrix} (m_1+m_2)gl_1\cos\theta_1 & 0 \\ 0 & mgl_2\cos\theta_2 \end{pmatrix}$$

$$\frac{\partial^2 V}{\partial \theta_2^2} = mgl_2\cos\theta_2$$

$$\partial^2 V(0,0) = \begin{pmatrix} (m_1+m_2)gl_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix} \to$$ MATRICE DEF. POSITIVA $\Rightarrow (0,0)$ è STAB.

$$\partial^2 V(0,\pi) = \begin{pmatrix} (m_1+m_2)gl_1 & \\ & -m_2 g l_2 \end{pmatrix} \to$$ NON DEF POSIT. $\Rightarrow (0,\pi)$ è INSTAB

$\partial^2 V(\pi,0)$ NON DEF. POS. $\Rightarrow$ $(\pi,0)$ è INSTAB.

$\partial^2 V(\pi,\pi)$ DEF. NEGA. $\Rightarrow$ NON DEF. POS. $\Rightarrow$ (π,π) è INSTAB.

4) Calcolare la FREQUENZA DELLE PICCOLE OSCILLAZIONI attorno al pto di eq. STAB.

$$\hat{\mathcal{L}} = \frac{1}{2}(\dot\theta_1,\dot\theta_2)\, A \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} - \frac{1}{2}(\theta_1,\theta_2)\, B \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$A = Q(0,0) = \begin{pmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix}$$

$$B = \partial^2 V(0,0) = \begin{pmatrix} (m_1+m_2)gl_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix}$$

Freq. piccole oscill. $\omega^2 = \lambda$ dove λ t.c.

$$0 = \det(B - \lambda A) =$$

$$= \det \begin{pmatrix} (m_1+m_2)gl_1 - \lambda(m_1+m_2)l_1^2 & -\lambda m_2 l_1 l_2 \\ -\lambda m_2 l_1 l_2 & m_2 g l_2 - \lambda m_2 l_2^2 \end{pmatrix}$$

$$= \det \begin{pmatrix} (m_1+m_2)l_1(g-\lambda l_1) & -\lambda m_2 l_1 l_2 \\ -\lambda m_2 l_1 l_2 & m_2 l_2 (g-\lambda l_2) \end{pmatrix}$$

$$0 = (m_1+m_2)\cancel{m_2}\cancel{l_1}\cancel{l_2}\,(g-\lambda l_1)(g-\lambda l_2) - \lambda^2 m_2^{\cancel{2}} l_1^{\cancel{2}} l_2^{\cancel{2}}$$

$$= \lambda^2 l_1 l_2 \left(m_1 + \cancel{m_2} - \cancel{m_2} \right) - (m_1+m_2)\, g\, (l_1+l_2)\lambda + (m_1+m_2) g^2$$

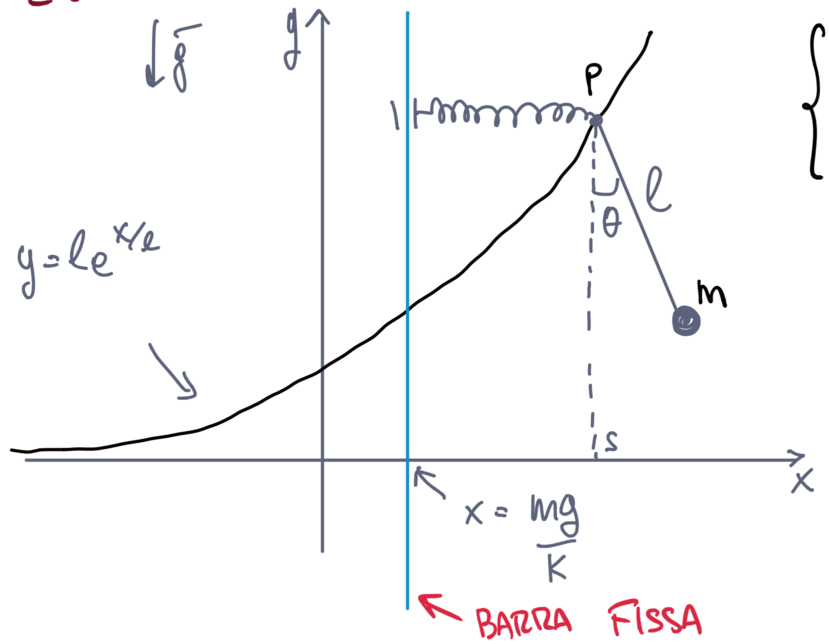
$$\lambda_{1,2} = \frac{(m_1+m_2)g(l_1+l_2)}{2 l_1 l_2 m_1} \pm \frac{1}{2 l_1 l_2 m_1}\sqrt{(m_1+m_2)^2 g^2 (l_1+l_2)^2 - 4(m_1+m_2)m_1 \cdot g^2 l_1 l_2}$$

$$= \frac{(m_1+m_2)(l_1+l_2)\, g}{2 m_1 \cdot l_1 \cdot l_2}\left[1 \pm \sqrt{1 - \frac{4\, \overbrace{m_1}^{<1}\, l_1 l_2}{\underbrace{(m_1+m_2)}\,(l_1+l_2)^2}}\;\right]$$

$\omega^2_{1,2} > 0$ (pto eq. STAB.)

$$\frac{4 l_1 l_2}{(l_1+l_2)^2} = \frac{(l_1+l_2)^2 - (l_1-l_2)^2}{(l_1+l_2)^2} < 1$$

ES. 2 del 25.09.19



$$\begin{cases} x = s + l \,\mathrm{sen}\,\theta \\ y = l e^{s/l} - l\cos\theta \end{cases}$$

$n = 2$

$\bar{q} = (s, \theta)$

1) Lagrangiana

$$\begin{cases} \dot{x} = \dot{s} + l\dot{\theta}\cos\theta \\ \dot{y} = \dot{s}\, e^{s/l} + l\dot{\theta}\,\mathrm{sen}\,\theta \end{cases}$$

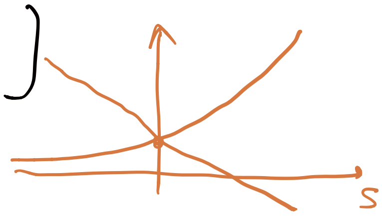
$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \Big(\dot{s}^2 + l^2\dot{\theta}^2\cos^2\theta + 2l\dot{s}\dot{\theta}\cos\theta + \dot{s}^2 e^{2s/l} + l^2\dot{\theta}^2\,\mathrm{sen}^2\theta + 2l\dot{s}\dot{\theta}\, e^{s/l}\,\mathrm{sen}\,\theta \Big)$$

$$= \frac{1}{2} m \left[\dot{s}^2 \left(1 + e^{2s/l}\right) + 2l\dot{s}\dot{\theta}\left(\cos\theta + e^{s/l}\,\mathrm{sen}\,\theta\right) + l^2\dot{\theta}^2 \right]$$

$$V = mgy + \frac{1}{2} K \left(x_P - \frac{mg}{K}\right)^2 =$$

$$= mgl\left(e^{s/l} - \cos\theta\right) + \frac{K}{2}\left(s - \frac{mg}{K}\right)^2$$

$$L = \frac{1}{2} m \left[\dot{s}^2 \left(1 + e^{2s/l}\right) + 2l\dot{s}\dot{\theta}\left(\cos\theta + e^{s/l}\,\mathrm{sen}\,\theta\right) + l^2\dot{\theta}^2 \right] - mgl\left(e^{s/l} - \cos\theta\right) - \frac{K}{2}\left(s - \frac{mg}{K}\right)^2 \Big]$$



3) Pti equil:

$$\frac{\partial V}{\partial s} = mg\, e^{s/l} + K\left(s - \frac{mg}{K}\right) = 0 \to e^{s/l} = 1 - \frac{sK}{mg}$$

$$\to s = 0$$

$$\frac{\partial V}{\partial \theta} = mgl \sin\theta \qquad = 0 \quad \longrightarrow \quad \theta = 0, \pi$$

2 pti di equil: $\quad (s^*, \theta^*) = (0,0) \ , \ (0,\pi)$

$$\partial^2 V = \begin{pmatrix} \frac{mg}{l} e^{s/l} + K & 0 \\ 0 & mgl\cos\theta \end{pmatrix}$$

$$\partial^2 V(0,0) = \begin{pmatrix} \frac{mg}{l} + k & 0 \\ 0 & mgl \end{pmatrix}$$ def. pos. $\Rightarrow$ $(0,0)$ è STAB.

$$\partial^2 V(0,\pi) = \begin{pmatrix} \frac{mg}{l} + k & 0 \\ 0 & -mgl \end{pmatrix}$$ non-def. pos. $\Rightarrow$ $(0,\pi)$ è INSTAB.

4) Linearizz. di L attorno a $(0,0)$.

$\rightarrow$ calcolo $A = a(0,0) \qquad B = \partial^2 V(0,0)$

$\rightarrow \hat{L} = \frac{1}{2}(\dot s \ \dot\theta) A \begin{pmatrix} \dot s \\ \dot\theta \end{pmatrix} - \frac{1}{2}(s \ \theta) B \begin{pmatrix} s \\ \theta \end{pmatrix}$

$$L = \frac{1}{2} m \left[\dot s^2 (1 + e^{2s/l}) + 2l \dot s \dot\theta (\cos\theta + e^{s/l}\sin\theta) + l^2 \dot\theta^2 \right] - mgl(e^{s/l} - \cos\theta) - \frac{k}{2}\left(s - \frac{mg}{k}\right)^2 \Big]$$

$\rightarrow$ espandere L in $s, \theta, \dot s, \dot\theta$ piccoli

$$\hat{L} = \frac{1}{2} m \Big[\dot s^2 \big(1 + 1 + \overset{\text{grado } 3}{\cancel{\tfrac{2s}{l}}} + \dots\big) + 2l \dot s \dot\theta \big(1 - \overset{\text{grado } 4}{\cancel{\tfrac{\theta^2}{2}}} + \dots + \overset{\text{grad} \geq 3}{\cancel{\big(1 + \tfrac{s}{l} + \dots\big)(\theta + \dots)}}\big) + l^2 \dot\theta^2 \Big]$$

$$- mgl\left(\cancel{1} + \frac{s}{l} + \frac{1}{2}\frac{s^2}{l^2} + \dots - \cancel{1} + \frac{\theta^2}{2} + \dots\right) - \frac{k}{2}\left(s^2 - 2\frac{mg}{k}s + \cancel{\frac{m^2g^2}{k^2}}\right)$$

trascuro tutti i termini di grado ≥ 3
e le costanti

$$= \frac{1}{2}m\left(2\dot{s}^2 + 2\ell\dot{s}\dot{\theta} + \ell^2\dot{\theta}^2\right) - \frac{mg}{2\ell}s^2 - \frac{k}{2}s^2 - \frac{mg\ell}{2}\theta^2$$

$$-\frac{1}{2}\left(\frac{mg}{\ell}+k\right)s^2 - \frac{mg\ell}{2}\theta^2$$

$$= \frac{1}{2}(\dot{s}\ \dot{\theta})\underbrace{\begin{pmatrix} 2m & m\ell \\ m\ell & m\ell^2 \end{pmatrix}}_{=A}\begin{pmatrix}\dot{s}\\ \dot{\theta}\end{pmatrix} - \frac{1}{2}(s\ \theta)\underbrace{\begin{pmatrix} \overset{\frac{2mg}{\ell}}{\frac{mg}{\ell}+k} & 0 \\ 0 & mg\ell \end{pmatrix}}_{=B}\begin{pmatrix}s\\ \theta\end{pmatrix}$$

5) Piccole oscillaz. quando $k = \frac{mg}{\ell}$

$$\det(B-\lambda A) = \det m\begin{pmatrix} \frac{2g}{\ell} - 2\lambda & -\ell\lambda \\ -\ell\lambda & g\ell - \ell^2\lambda \end{pmatrix} =$$

$$= m^2\left\{2\ell^2\left(\frac{g}{\ell}-\lambda\right)\left(\frac{g}{\ell}-\lambda\right) - \ell^2\lambda^2\right\} =$$

$$= m^2\ell^2\left\{2\left(\lambda^2 - 2\lambda\frac{g}{\ell} + \left(\frac{g}{\ell}\right)^2\right) - \lambda^2\right\} =$$

$$= m^2\ell^2\left\{\lambda^2 - 4\frac{g}{\ell}\lambda + 2\left(\frac{g}{\ell}\right)^2\right\} = 0$$

$$\lambda_{1/2} = \frac{2g}{\ell} \pm \sqrt{4\left(\frac{g}{\ell}\right)^2 - 2\left(\frac{g}{\ell}\right)^2}$$

$$= \frac{g}{\ell}\left[2 \pm \sqrt{2}\right] > 0$$

6) Calcolare i modi normali di oscillaz.

↳ trovare autovett. $\bar{u}$, cioè

$$(B-\lambda A) = \left(\begin{matrix} \frac{2g}{\ell} - 2\lambda & -\ell\lambda \\ -\ell\lambda & g\ell - \ell^2\lambda \end{matrix}\right)$$

$$(B - \lambda_i A)\,\bar{u}^{(i)} = 0$$

$$\lambda_1 = (2+\sqrt{2})\,\frac{g}{\ell} \quad : \quad \begin{pmatrix} \frac{2g}{\ell}(-1-\sqrt{2}) & -g(2+\sqrt{2}) \\ -g(2+\sqrt{2}) & g\ell(-1-\sqrt{2}) \end{pmatrix} \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = 0$$

$$\rightarrow \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \end{pmatrix} = \begin{pmatrix} \ell\,(2+\sqrt{2}) \\ -2\,(1+\sqrt{2}) \end{pmatrix}$$

$$\lambda_2 = (2-\sqrt{2})\,\frac{g}{\ell} \rightarrow \begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \end{pmatrix} = \begin{pmatrix} \ell\,(2-\sqrt{2}) \\ -2(1-\sqrt{2}) \end{pmatrix}$$

⇝ modi normali:

$$\begin{pmatrix} s(t) \\ \theta(t) \end{pmatrix}^{(1)} = A_1\,\bar{u}^{(1)} \cos\left(\sqrt{\lambda_1}\,t + \varphi_1\right)$$

$$\begin{pmatrix} s(t) \\ \theta(t) \end{pmatrix}^{(2)} = A_2\,\bar{u}^{(2)} \cos\left(\sqrt{\lambda_2}\,t + \varphi_2\right)$$

$A_1, A_2 \ll 1$

⇓

il moto armonico approssima bene il sist. non-lineare di partenza