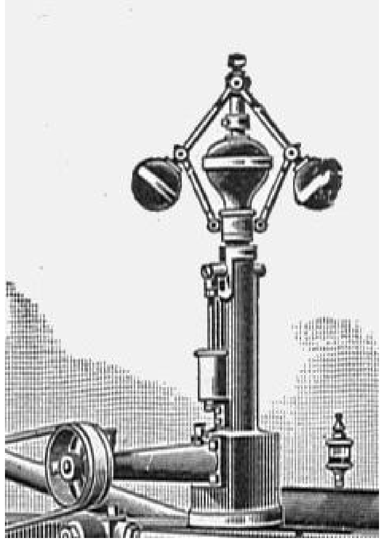
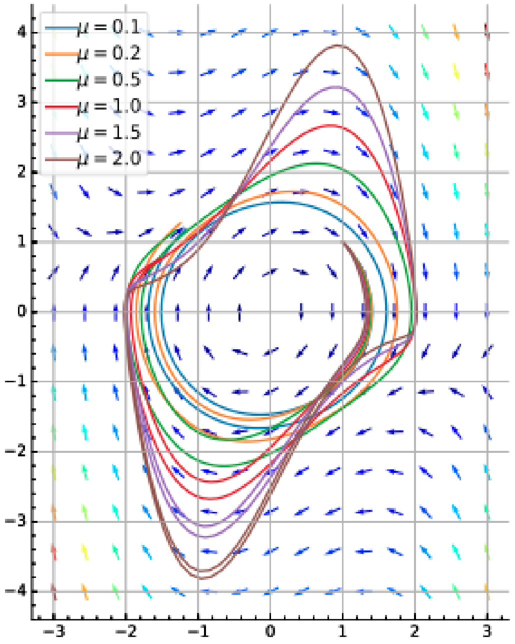
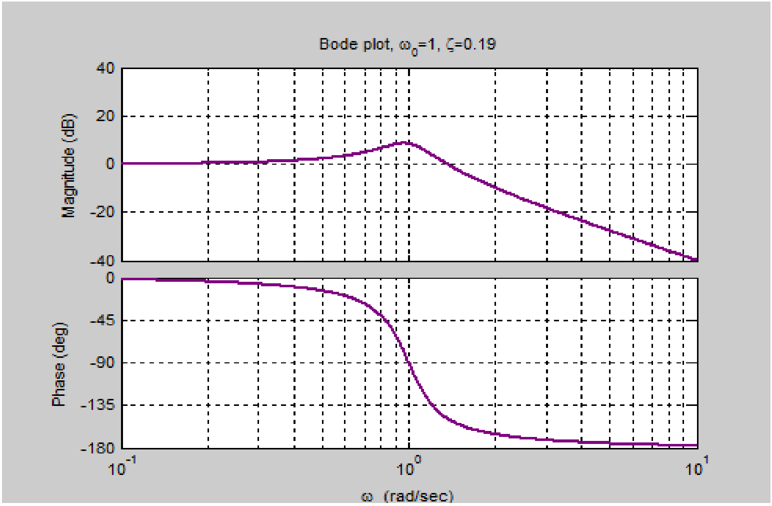


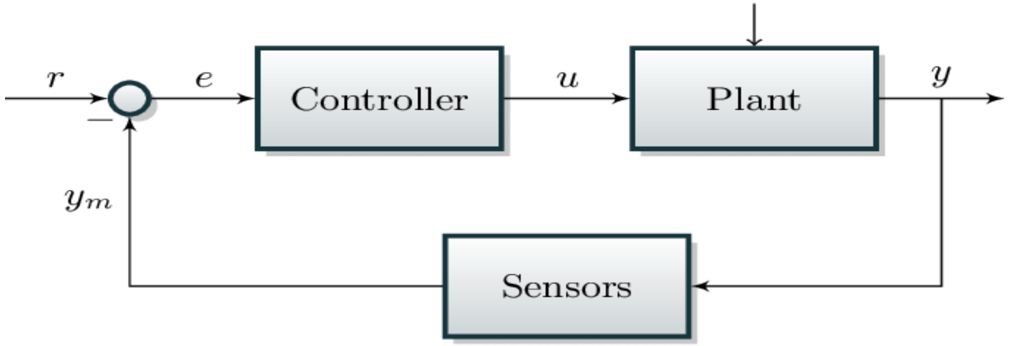


Introduction to **Control Systems**

Theory and applications

Enrico Regolin / Laura Nenzi





Course Overview (1)

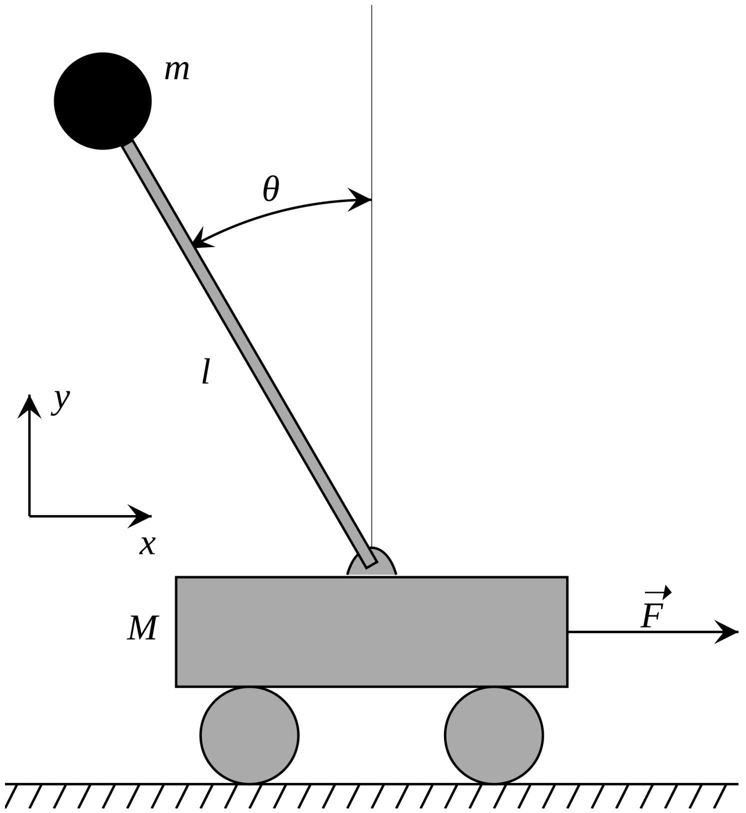
- Day 1: Linear Control (time domain)
  - Introduction
  - Dynamical Linear Systems
  - Observability & Controllability
  - PID Controllers
  - Luenberger Observer

- Day 2: Linear Control (frequency domain)
  - From State-space to Transfer Function
  - Classic Control Elements (Bode Diagram / Root Locus)
  - Introduction to Simulink.
  - Ctrl Lab (days 1,2)

Course Overview (2)

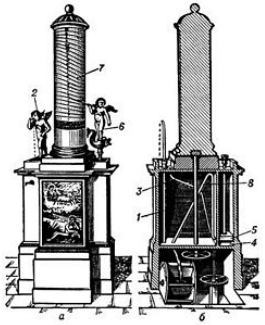
- Day 3: Optimal Control and KF Estimation
  - Optimal Control (LQR)
  - Model Predictive Control
  - Kalman Filtering
  - Sliding Mode Control (tentative)

- Day 4: Control Laboratory
  - Kalman Filtering and Optimal Control
  - Matlab/Simulink
  - Cart-pole



Control Systems History

- Water Clock
  - Alexandria
    (Ctesibius, 3rd century BC)



- Centrifugal Governor
  - Windmills
    (C. Huygeens, 17th century)
  - Steam Engine
    (J. Watt, 1788)

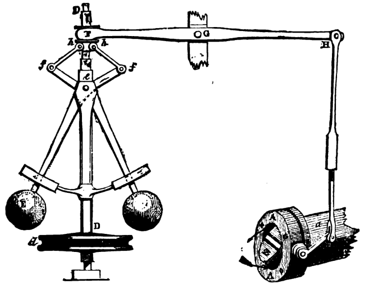
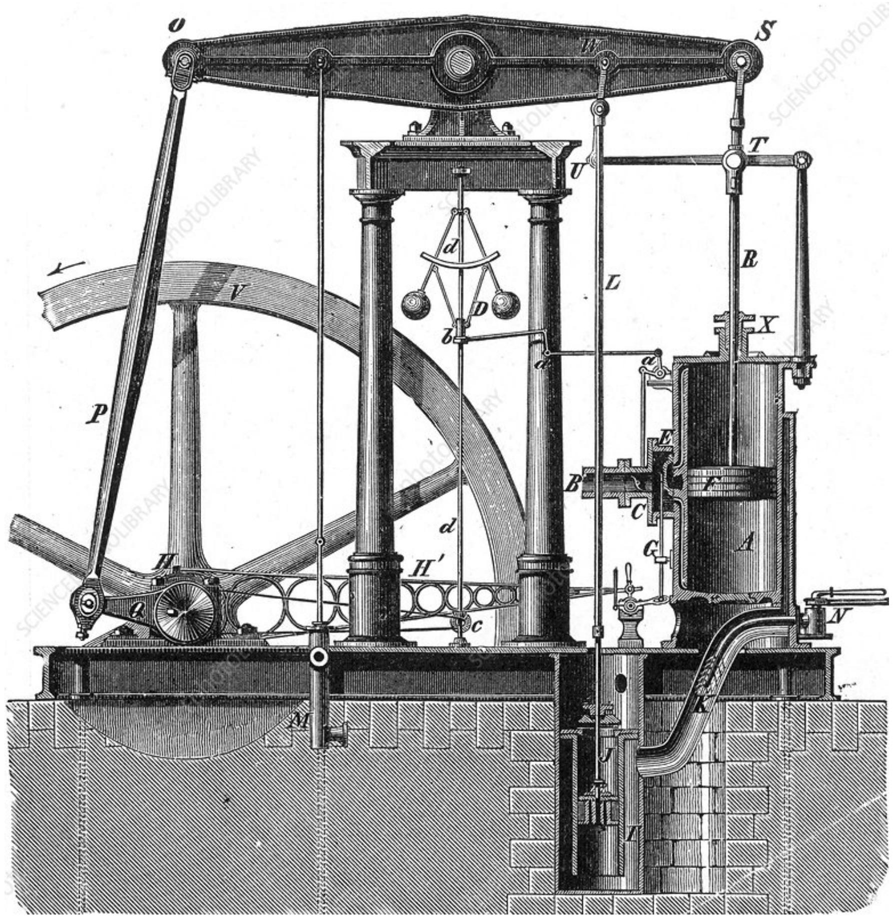
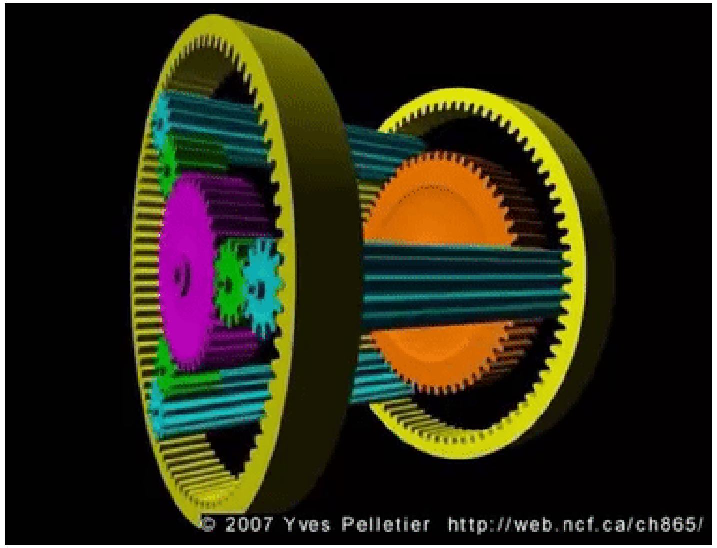


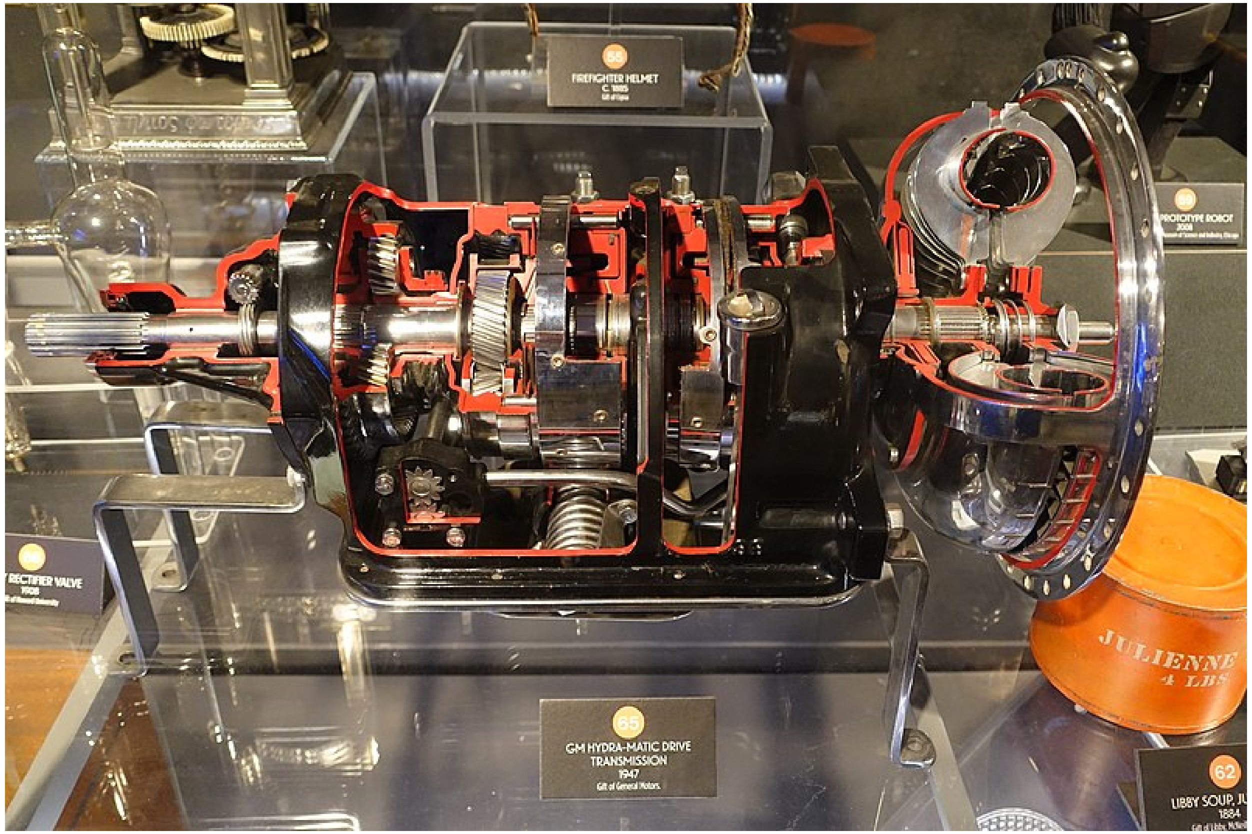
FIG. 4.—*Governor and Throttle-Valve.*



Control Systems History

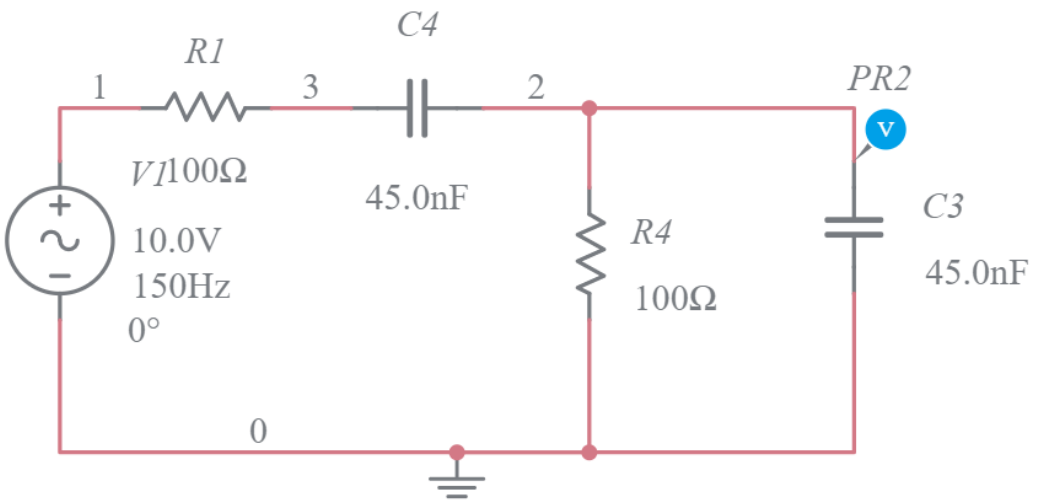
- First Automatic Transmission (Hydramatic, 1939)

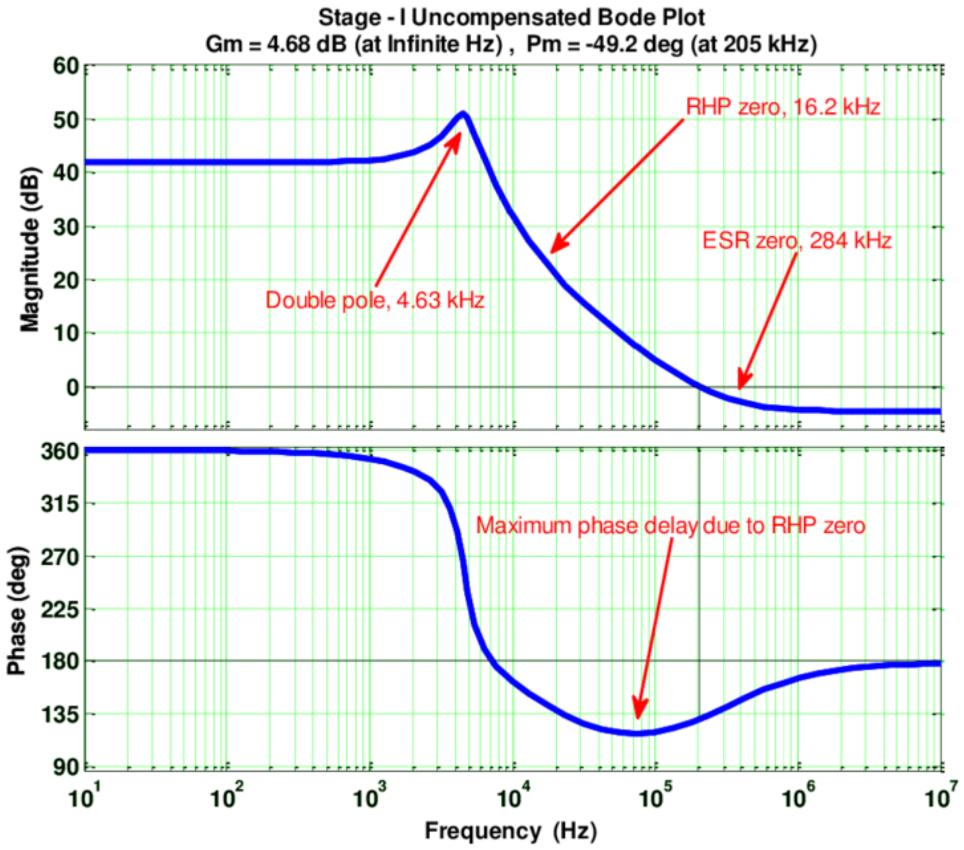




Control Systems History

- Classical control theory formalized from circuits theory





Tacoma Bridge Collapse

Day 1

Linear Control (time domain)

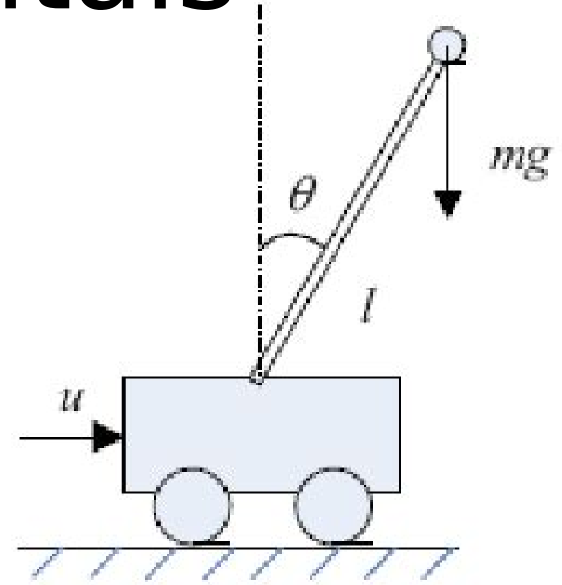
Control Systems Fundamentals

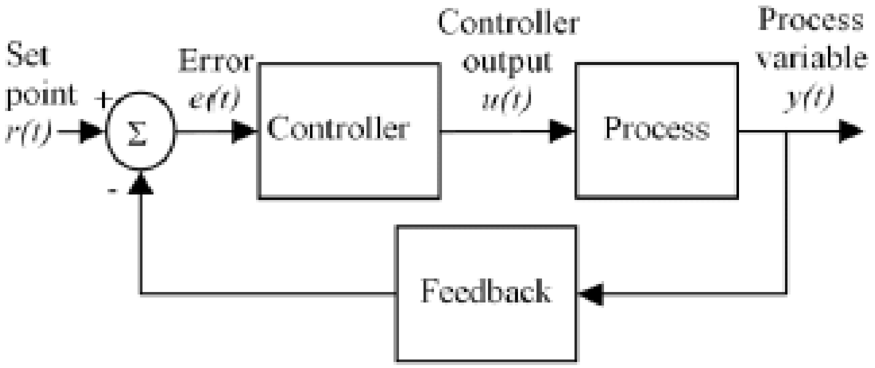
REQUIRED

- Dynamical System MODEL
- Control Input (non-autonomous systems)
- Reference Signal

CHALLANGES

- Missing/Noisy Information
- Physical limitations





Dynamical Systems (1)

Past history (state) influences future output

- **Continuous Time** **vs.** **Discrete Time**

$$\dot{x} = f(x), \quad t \in [0, \infty)$$

$$x(k+1) = f(x(k)), \quad k = 0, 1, 2, \ldots$$

- **Autonomous** **vs.** **Non-autonomous**

$$\dot{x} = f(x)$$

$$\dot{x} = f(x, u)$$

- **Linear** **vs.** **Non-linear**

$$\dot{x}_1 = -2x_2$$
$$\dot{x}_2 = 0.5x_1 + x_2 + 0.4u$$

$$\dot{x}_1 = -x_1 x_2$$
$$\dot{x}_2 = 0.5x_1^2 + \sin(x_2) + \frac{0.4}{u}$$

Dynamical Systems (2)

- **SISO** **vs.** **MIMO**

$$\dot{x} = Ax + b \cdot u$$

$$y = Cx (= 0.5x_1)$$

$$\dot{x} = Ax + B\mathbf{u}$$

$$\mathbf{y} = Cx$$

- **Time Invariant** **vs.** **Time Variant**

$$\dot{x} = f(x, u)$$

$$\dot{x} = Ax + Bu$$

$$\dot{x}(t) = f(x(t), u(t), t)$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

- **Deterministic** **vs.** **Non-Deterministic (Stochastic, noisy, etc.)**

$$\dot{x} = -x^2 - x + u$$

$$y = 0.5x$$

$$x(k+1) = -(2+\nu)x(k)^2 - x(k) + u(k)$$

$$y(k) = 0.5x(k) + \eta$$

$$\nu \sim N(\mu, \sigma), \eta \sim U(0, 1)$$

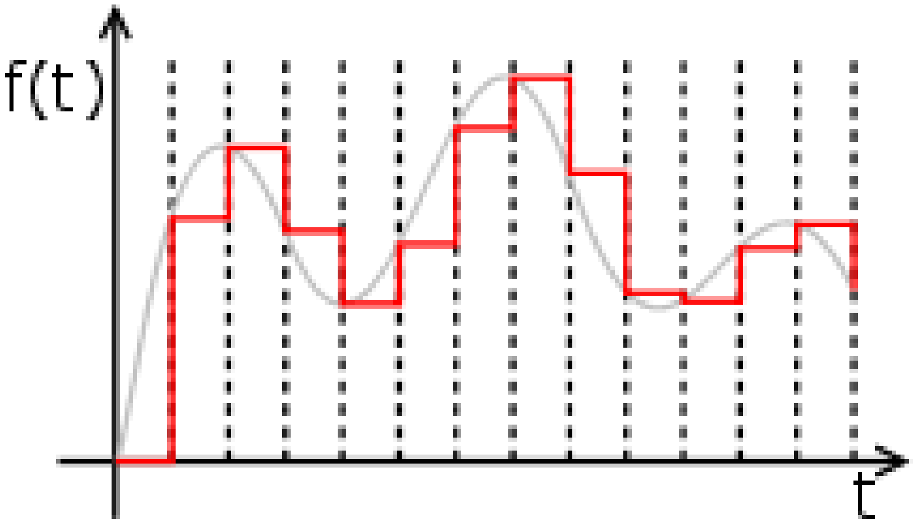
Dynamical Systems (3)

- **LTI systems --- State-Space representation** $x(0) = x_0,\ x \in \mathbb{R}^n$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$A_d = e^{A\Delta T}$$
$$B_d = A^{-1}(e^{A\Delta T} - 1)B$$

$$x(k+1) = A_d x(k) + B_d u(k)$$
$$y(k) = Cx(k) + Du(k)$$



Dynamical Systems (3)

- **LTI systems --- State-Space representation** $x(0) = x_0, \; x \in \mathbb{R}^n$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$A_d = e^{A\Delta T}$$
$$B_d = A^{-1}(e^{A\Delta T} - 1)B$$

$$x(k+1) = A_d x(k) + B_d u(k)$$
$$y(k) = Cx(k) + Du(k)$$

- **Output response (continuous time)**

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Free Response (homogeneous solution)

Effect of input

Output response (discrete time)

$$y(k) = CA_d^k x_0 + C\sum_{i=0}^{k-1} A_d^{k-1-i} B_d u(i) + Du(k)$$

Stability condition (Hurwitz)

$$x(t) = e^{at}$$

$a < 0$ $a > 0$

$$real(eig(A)) < 0$$

$$x(k) = a^k$$

$|a| < 1$ $|a| > 1$

$$|eig(A_d)| < 1$$

State-Space Realizations

Similarity Transformations

- The choice of a state-space model for a given system is not unique.
- For example, let T be an invertible matrix, and consider a coordinate transportation $x = T\tilde{x}$, i.e., $\tilde{x} = T^{-1}x$. This is called a similarity transformation.
- The standard state-space model can be written as

$$\begin{cases} \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{cases} \Rightarrow \begin{cases} T\dot{\tilde{x}} &= AT\tilde{x} + Bu, \\ y &= CT\tilde{x} + Du. \end{cases}$$

  i.e.,

$$\begin{aligned} \dot{\tilde{x}} &= (T^{-1}AT)\tilde{x} + (T^{-1}B)u = \tilde{A}\tilde{x} + \tilde{B}u \\ y &= (CT)\tilde{x} + Du = \tilde{C}\tilde{x} + \tilde{D}u. \end{aligned}$$

- You can check that the time response is exactly the same for the two models (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$!

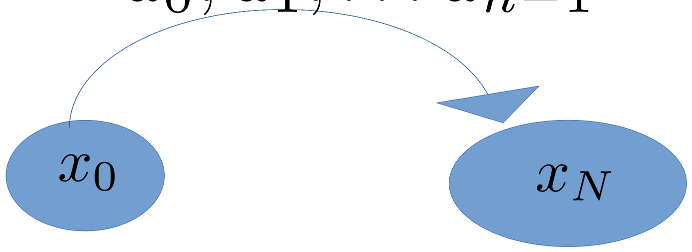
LTI Systems Properties

Discrete case

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

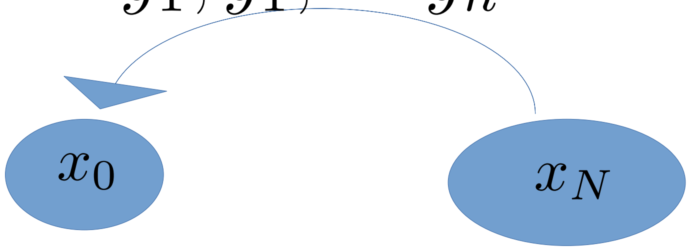
Reaching a state

$u_0, u_1, \dots u_{n-1}$



“Observing” the initial state

$y_1, y_1, \dots y_n$



LTI Systems Properties

Conditions for all LTI systems:

- <u>Controllability</u> $\iff rank(\mathcal{C}) = n$

$$\mathcal{C} = \left[B, AB, A^2B, \ldots, A^{n-1}B\right]$$

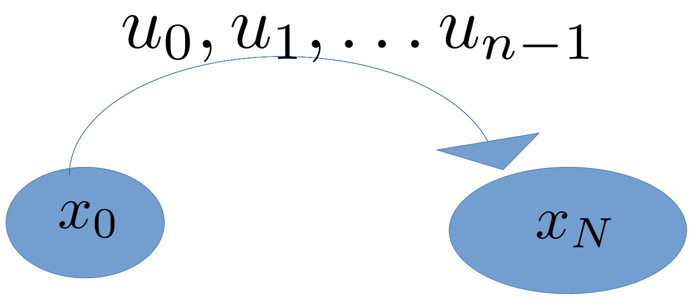
- <u>Observability</u> $\iff rank(\mathcal{O}) = n$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \ldots \\ CA^{n-1} \end{bmatrix}$$

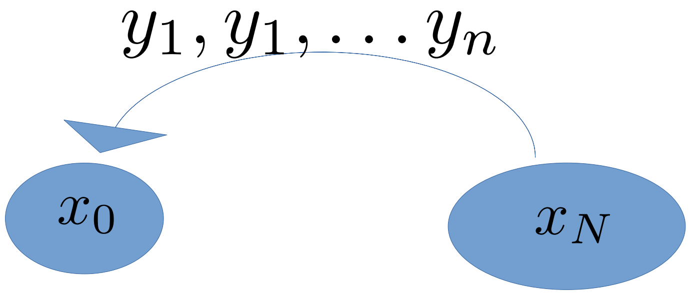
Discrete case

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

Reaching a state



“Observing” the initial state



LTI Systems Properties

- Pair (A,B) is "Controllable" $\Leftrightarrow rank(\mathcal{C}) = n$
- Pair (A,C) is "Observable" $\Leftrightarrow rank(\mathcal{O}) = n$
- LTI System $\mathcal{S} : \{A, B, C\}$ is a "minimal state-space realization" if it is both observable and controllable.

- Example:

$$\mathcal{S}_0 : \{A_0, B, C\}\,, \quad \mathcal{S}_1 : \{A_1, B, C\}$$

$$B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\mathcal{C}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \mathcal{O}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$rank(\mathcal{C}_0) = 1 \qquad rank(\mathcal{O}_0) = 2$$

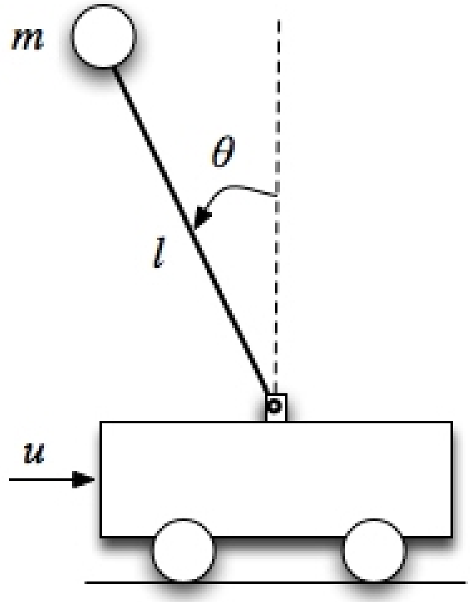
$$\mathcal{C}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \mathcal{O}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$rank(\mathcal{C}_1) = 3 \qquad rank(\mathcal{O}_1) = 3$$

non-LTI Systems (example)

Is the inverted pendulum (cartpole) controllable?

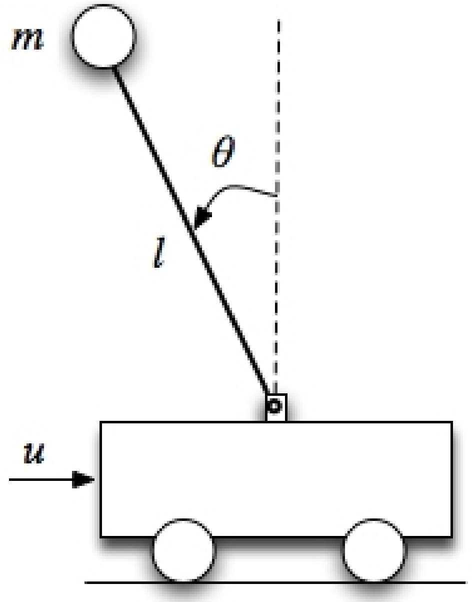
$$\begin{cases} \ddot{p} & = \frac{u + m\,l\,\dot{\theta}^2\ \sin\theta - m\,g\ \cos\theta \sin\theta}{M + m \sin\theta^2} \\ \ddot{\theta} & = \frac{g\ \sin\theta - \cos\theta \ddot{p}}{l} \end{cases}$$



In non-linear systems Controllability and Observability Matrices represent LOCAL properties.

non-LTI Systems (example)

Is the inverted pendulum (cartpole) controllable?



$$\begin{cases} \ddot{p} & = \frac{u+m\,l\,\dot{\theta}^2\ \sin\theta - m\,g\ \cos\theta\sin\theta}{M+m\sin\theta^2} \\ \ddot{\theta} & = \frac{g\ \sin\theta - \cos\theta\ddot{p}}{l} \end{cases}$$

In non-linear systems Controllability and Observability Matrices represent LOCAL properties.

$$\dot{x} = f(x,u), \quad \text{eq.point } x_0, u_0$$

$$\dot{x} = \underline{A}x + \underline{B}u$$

$$\underline{A} = \frac{\partial f(x,u)}{\partial x}_{|x=x_0,u=u_0}$$

$$\underline{B} = \frac{\partial f(x,u)}{\partial u}_{|x=x_0,u=u_0}$$

$$x = \left[p,\ \dot{p},\ \theta,\ \dot{\theta}\right]^T$$

$$\frac{\partial f}{\partial u} = \left[0,\ \frac{1}{(M+m(1-\cos^2(\theta))},\ 0,\ \frac{-\cos(\theta)}{L(M+m(1-\cos^2(\theta))}\right]^T$$

non-LTI Systems (example)

Linearization

$$\dot{x} = f(x,u), \quad \text{eq.point } x_0, u_0$$

$$\dot{x} = \underline{A}x + \underline{B}u$$

$$\underline{A} = \frac{\partial f(x,u)}{\partial x}|_{x=x_0,u=u_0}$$

$$\underline{B} = \frac{\partial f(x,u)}{\partial u}|_{x=x_0,u=u_0}$$

$$(\dot{x} = 0,\ \theta_0 = 0,\ \dot{\theta}_0 = 0,\ u_0 = 0)$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -gm/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \alpha & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/(Ml) \end{bmatrix}$$

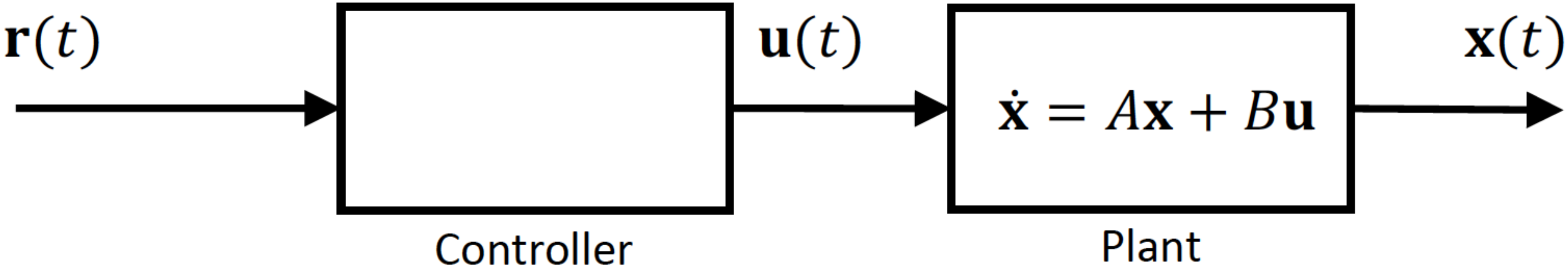
$$\alpha = \frac{(m+M)g}{Ml}$$

$$M = 1,\ m = 0.1,\ g = 9.81,\ l = 0.5$$

$$\mathcal{C} \approx \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -43 \\ -2 & 0 & -43 & 0 \end{bmatrix}$$

$$rank(\mathcal{C}) = 4$$

Reference Tracking



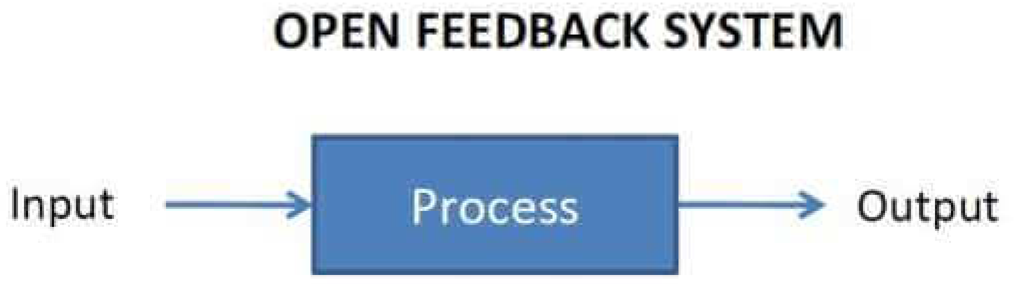
Control objectives:

- Reject disturbances (if there is some perturbation in state, making it get back to initial state)
- Follow reference trajectories (if we want the system to have a certain $\boldsymbol{x_{ref}}$)
- Make system follow some other “desired behavior”

Open-loop vs. Closed-loop

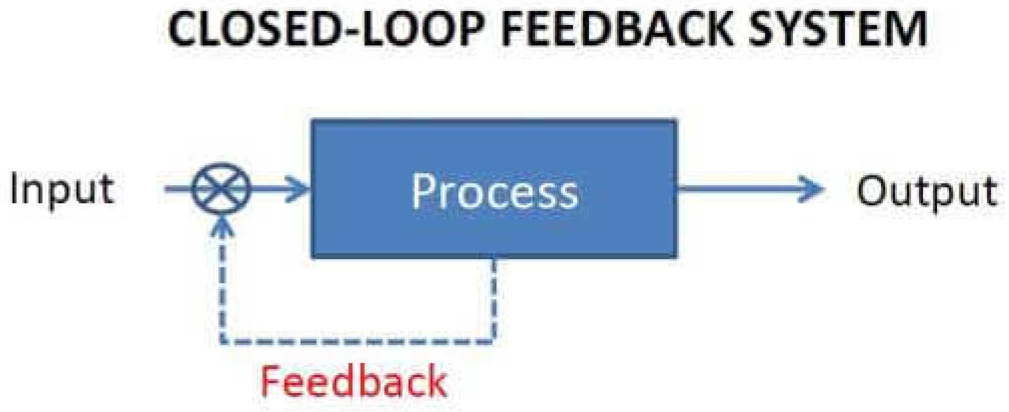
Open-loop or feed-forward control

- Control action does not depend on plant output
- Cheaper, no sensors required.
- Quality of control generally poor without human intervention

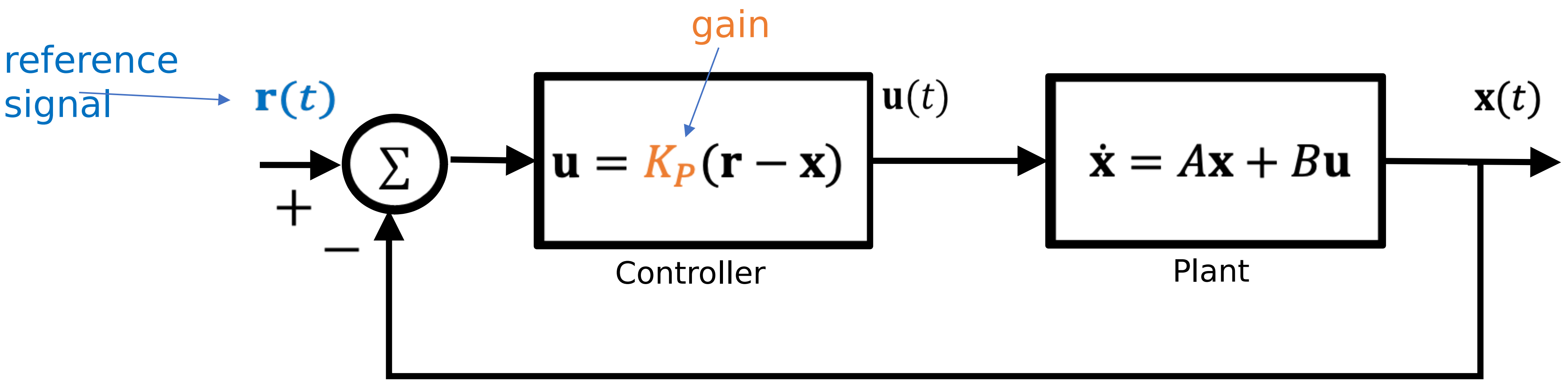


Feed-back control

- Controller adjusts controllable inputs in response to observed outputs
- Can respond better to variations in disturbances
- Performance depends on how well outputs can be sensed, and how quickly controller can track changes in output



Proportional Controller



- Common objective: make plant state *track* the reference signal $\mathbf{r}(t)$
- $e = r - x$ is the error signal
- Closed-loop dynamics: $\dot{\mathbf{x}} = A\mathbf{x} + BK_P(\mathbf{r} - \mathbf{x}) = (A - BK_P)\mathbf{x} + BK_P\mathbf{r}$
- pick K_P s.t. the composite system is asymptotically stable, i.e. pick K_P such that eigenvalues of $(A - BK)$ have negative real-parts

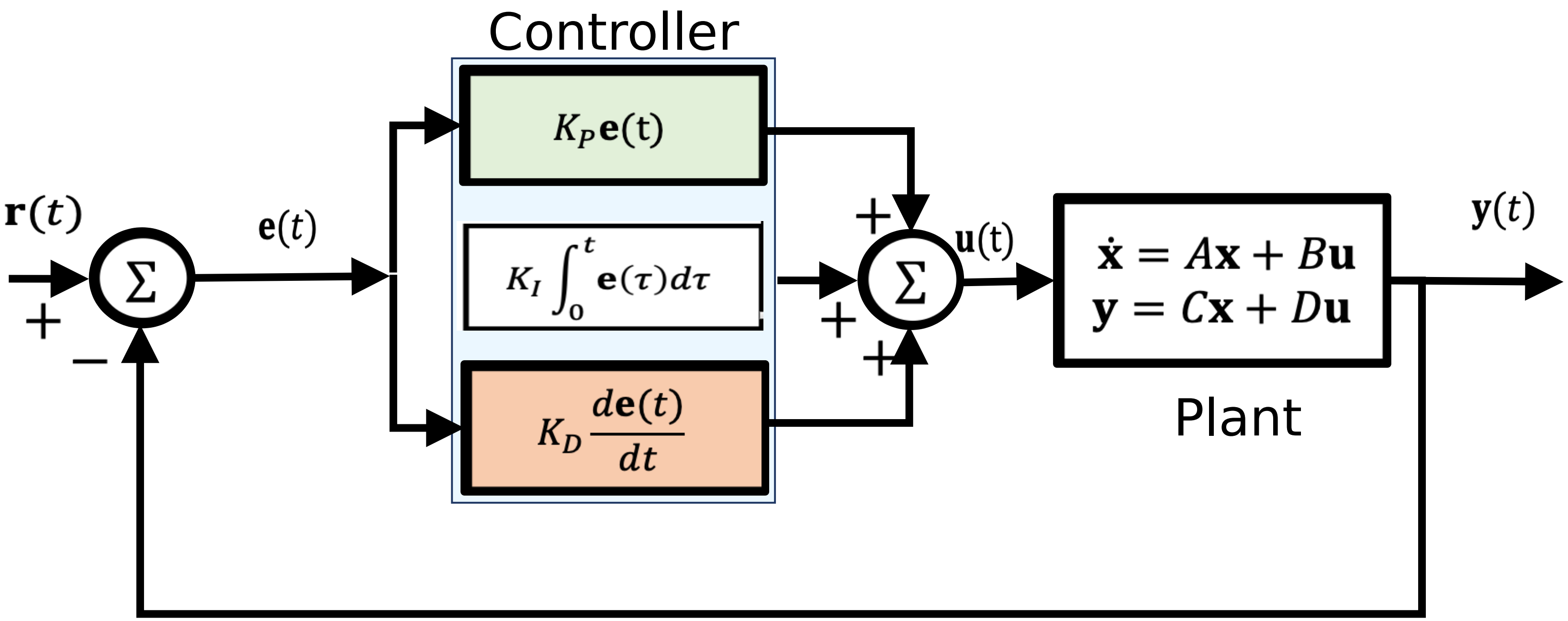
P. Ctrl: eigenvalues assignment

- Initial LTI system $$A = \begin{bmatrix} 4 & 6 \\ 1 & 3 \end{bmatrix},\ B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

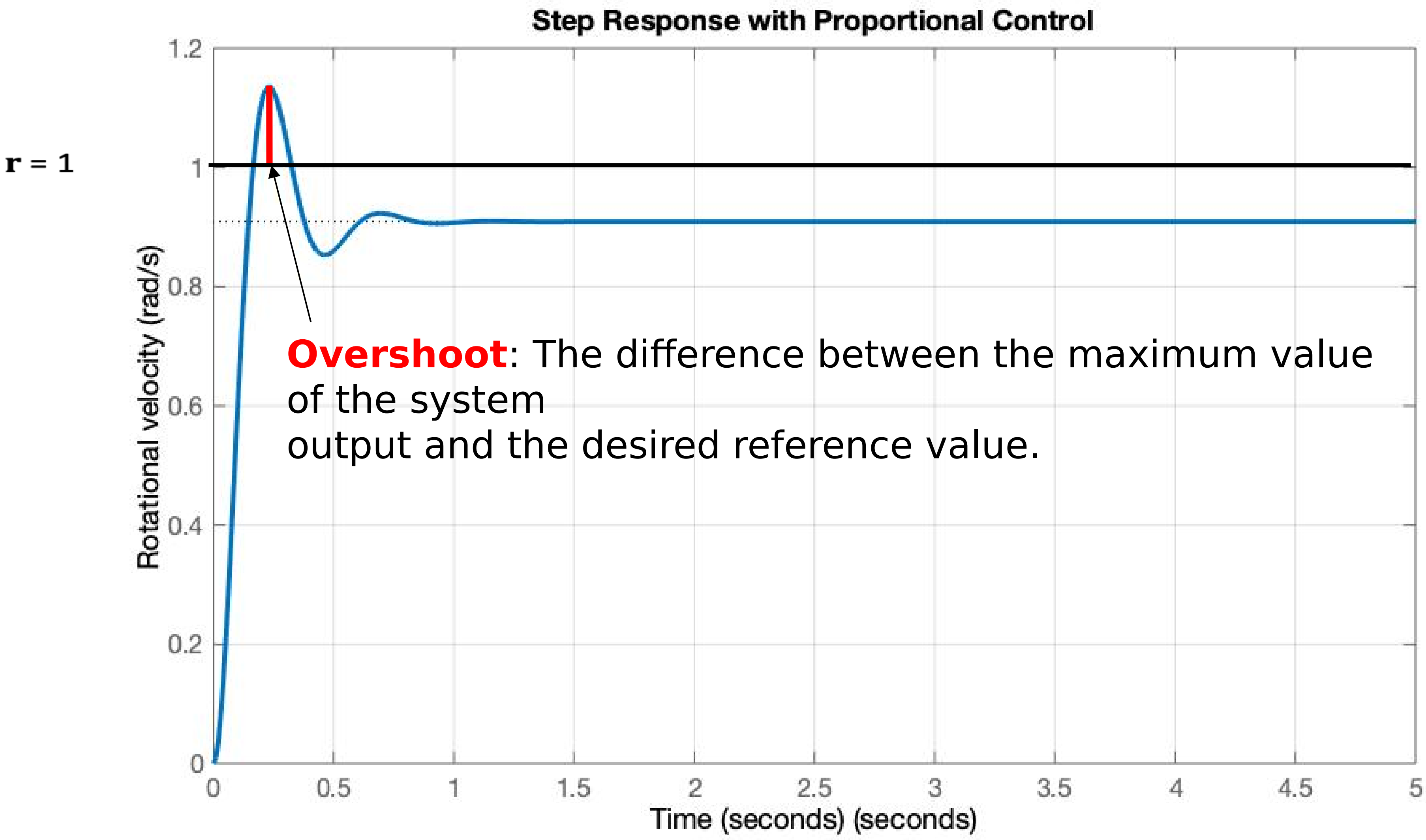
  - Note $\text{eigs}(A) = 6, 1 \Rightarrow$ unstable plant!
  - Let $K = (k_1 \quad k_2)$. Then, $A - BK = \begin{pmatrix} 4 - 2k_1 & 6 - 2k_2 \\ 1 - k_1 & 3 - k_2 \end{pmatrix}$
  - Solve the equation: $\det(A - BK - \lambda \boldsymbol{I})$=0, i.e. $\lambda^2 + (2k_1 + k_2 - 7)\lambda + (6 - 2k_2) = 0$
  - 2 distinct solution if polynomial of the form $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + (-\lambda_1 - \lambda_2)\lambda + \lambda_1 \lambda_2$
  - That means: $2k_1 + k_2 - 7 = (-\lambda_1 - \lambda_2)$ and $6 - 2k_2 = \lambda_1 \lambda_2$
  - $\lambda_1 = -1, \lambda_2 = -2$ gives $k_1 = 4,\ k_2 = 2$

Proportional Integral Derivative (PID) controllers

- Entire state in most cases is not available, feedback only based on **y**
- How do we evaluate the controlled system performance?

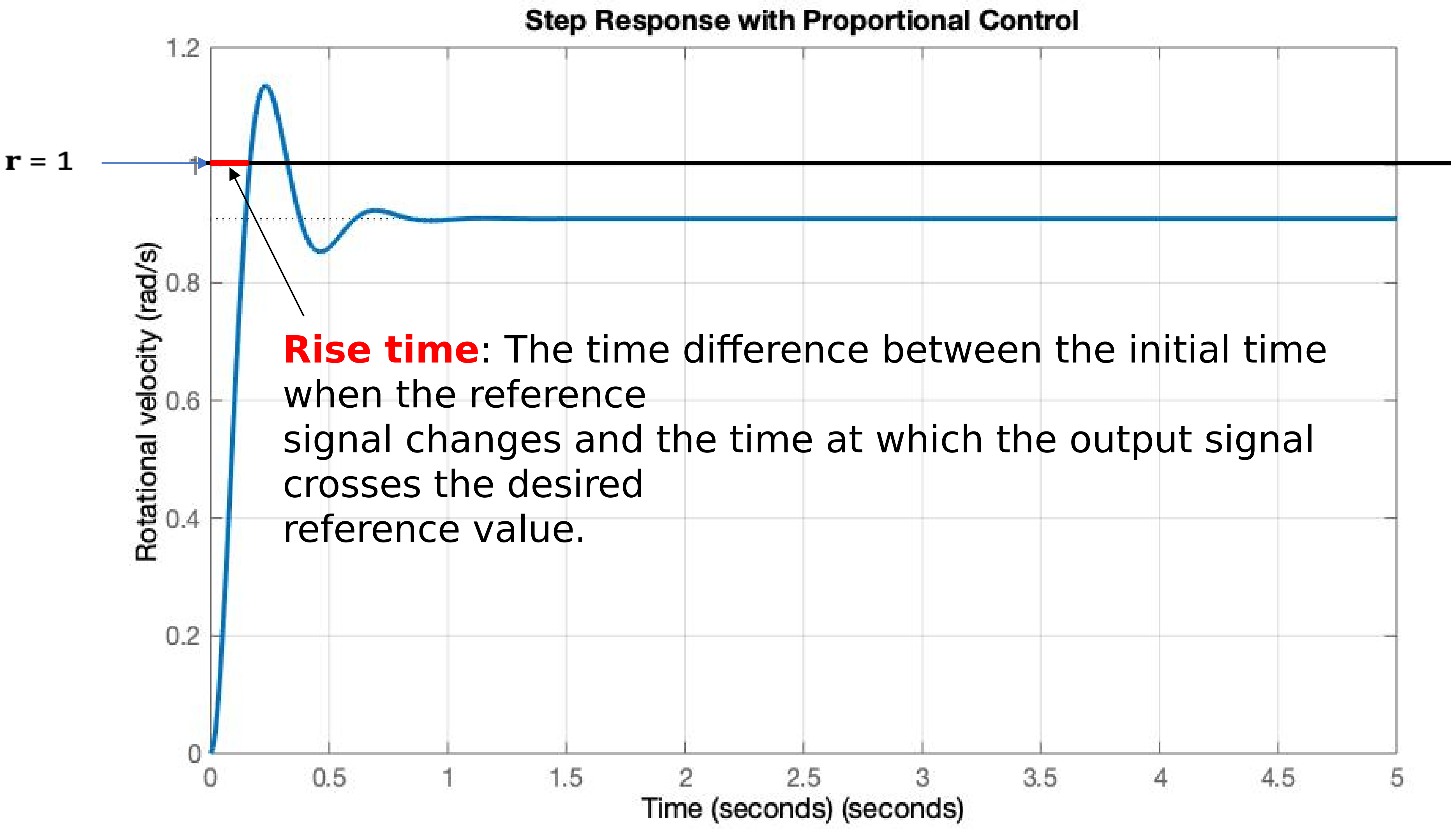


Measuring control performance



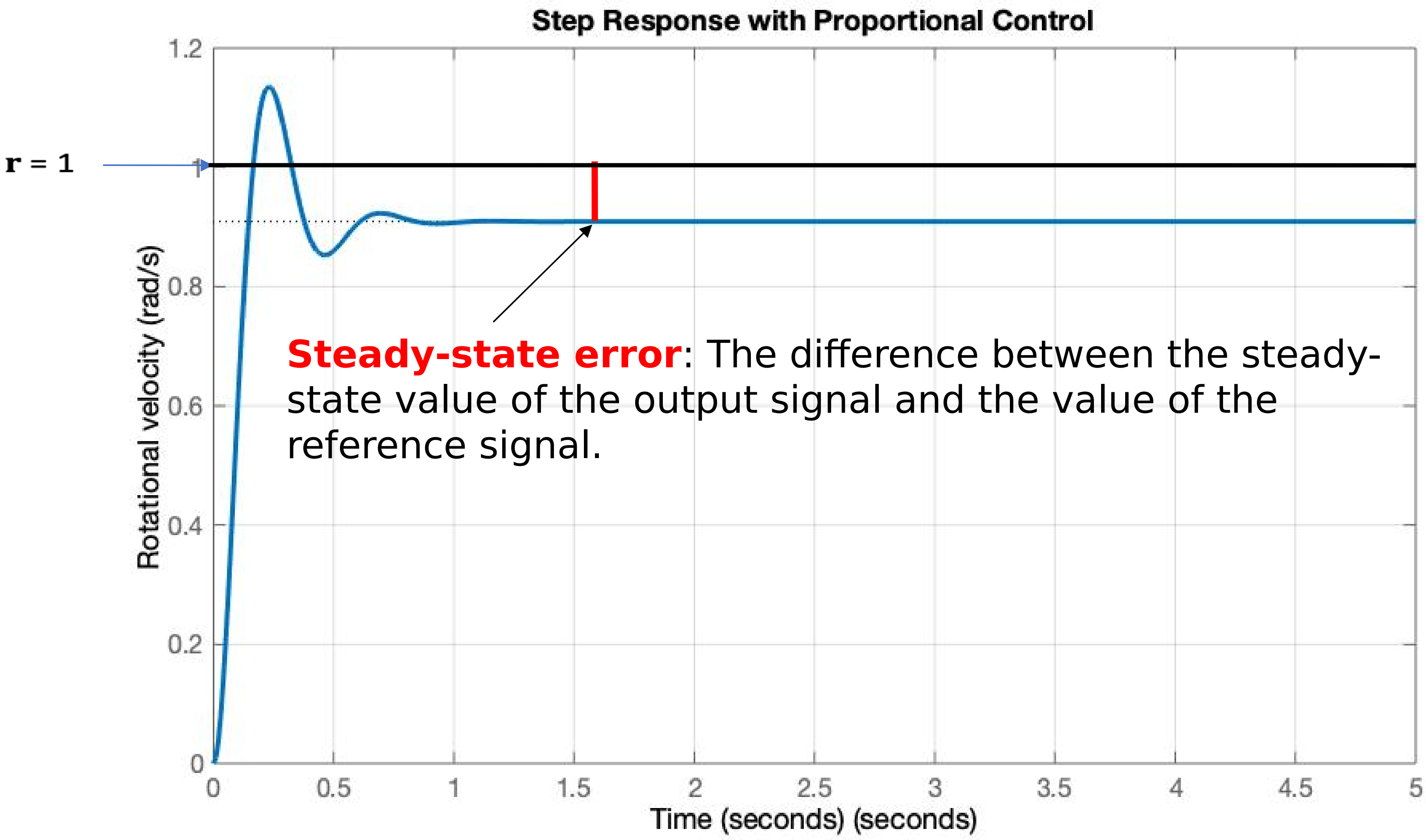
Overshoot: The difference between the maximum value of the system
output and the desired reference value.

Measuring control performance



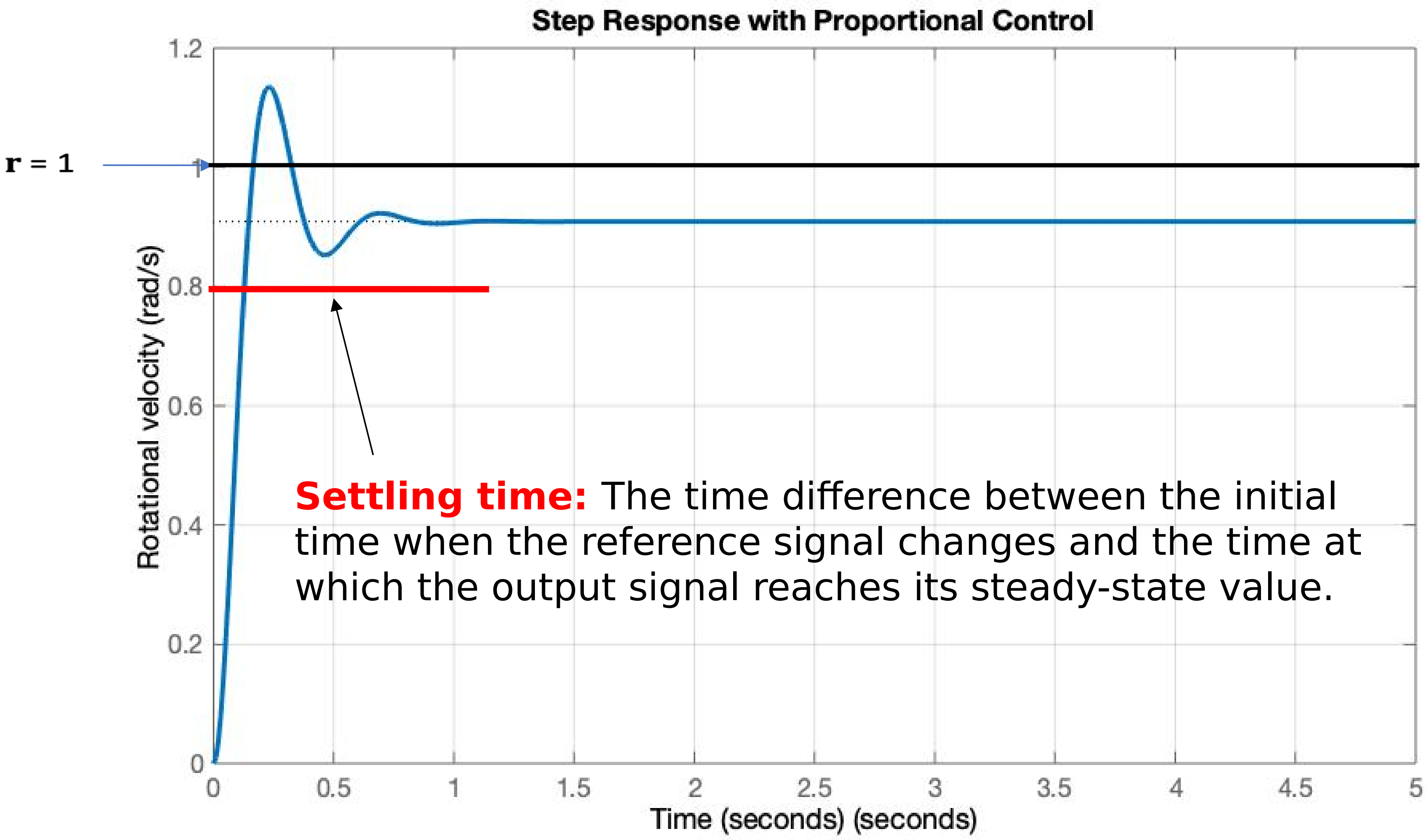
Rise time: The time difference between the initial time when the reference
signal changes and the time at which the output signal crosses the desired
reference value.

Measuring control performance



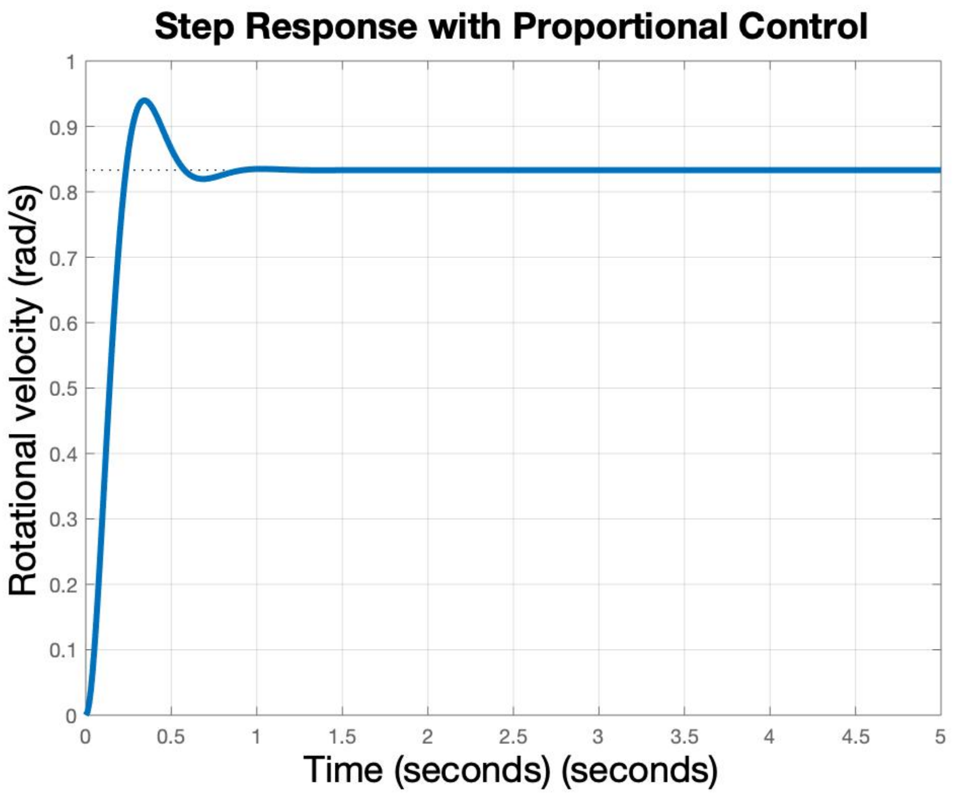
Steady-state error: The difference between the steady-state value of the output signal and the value of the reference signal.

Measuring control performance

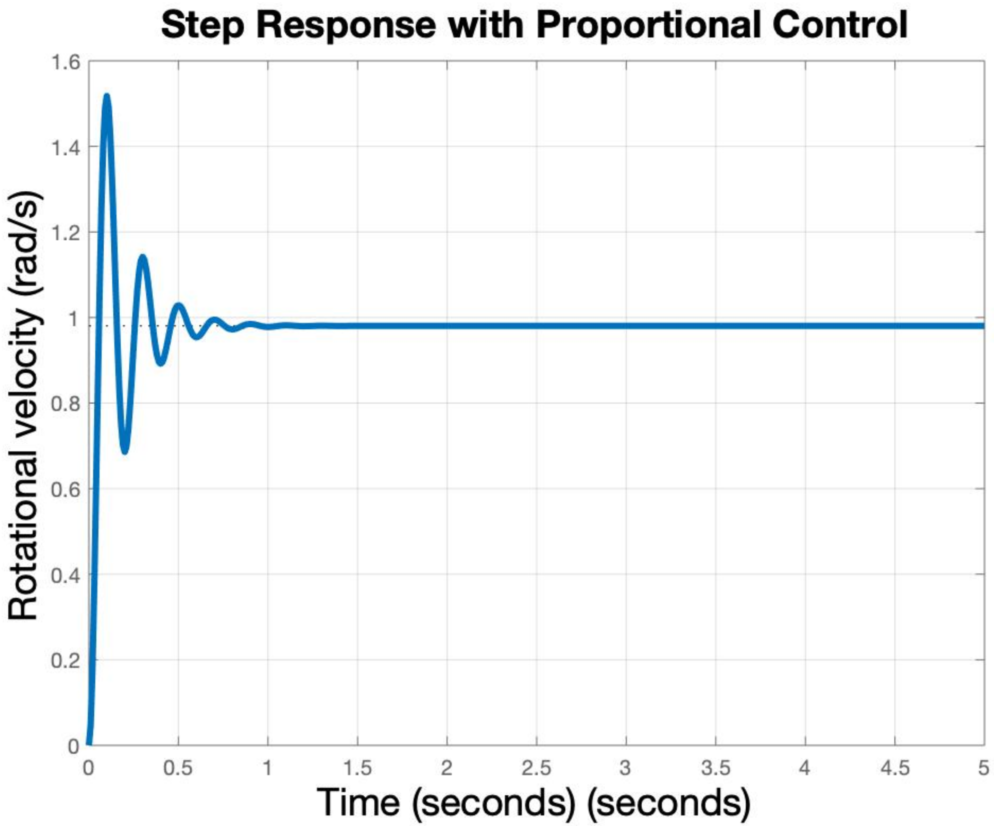


Settling time: The time difference between the initial time when the reference signal changes and the time at which the output signal reaches its steady-state value.

Measuring control performance



$\mathbf{K_P} = 50$



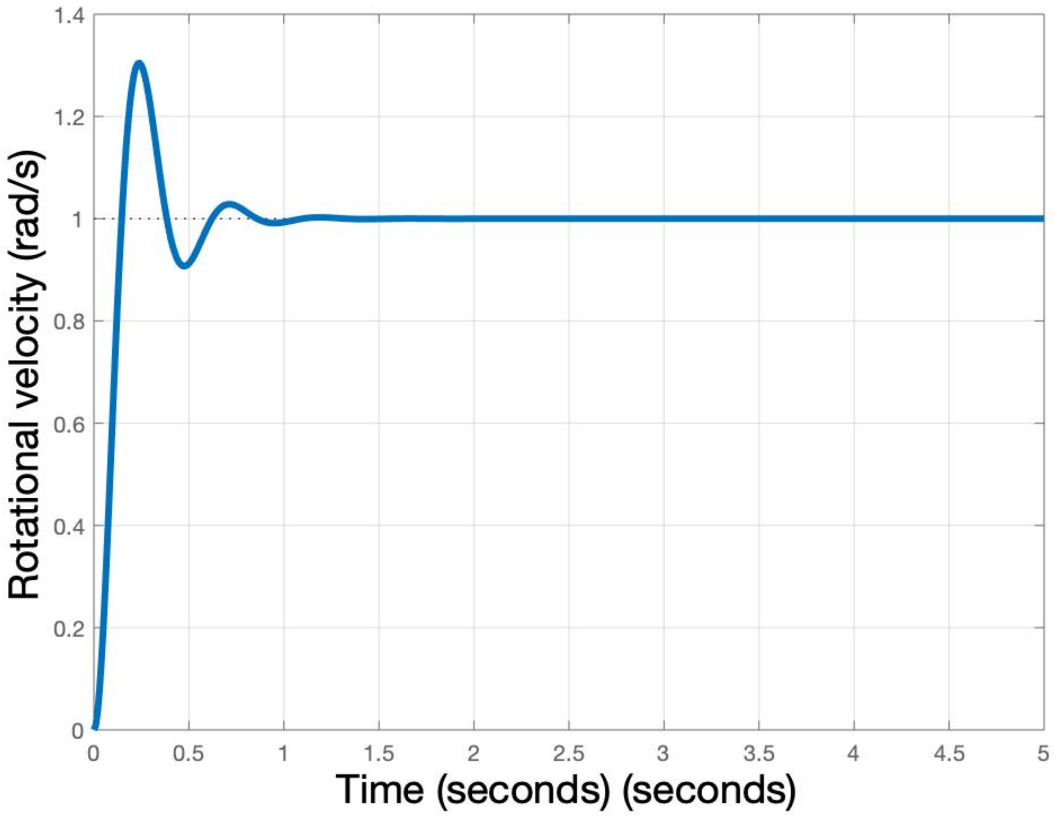
$\mathbf{K_P} = 500$

P-only controller

- Compute error signal $\mathbf{e} = \mathbf{r} - \boldsymbol{y}$
- Proportional term $K_p\mathbf{e}$:
  - K_p proportional gain;
  - Feedback correction proportional to error
- Cons:
  - If K_p is small, error can be large! [undercompensation]
  - If K_p is large,
    - system may oscillate (i.e. unstable) [overcompensation]
    - may not converge to set-point fast enough
  - P-controller always has steady state error or offset error

PI-controller

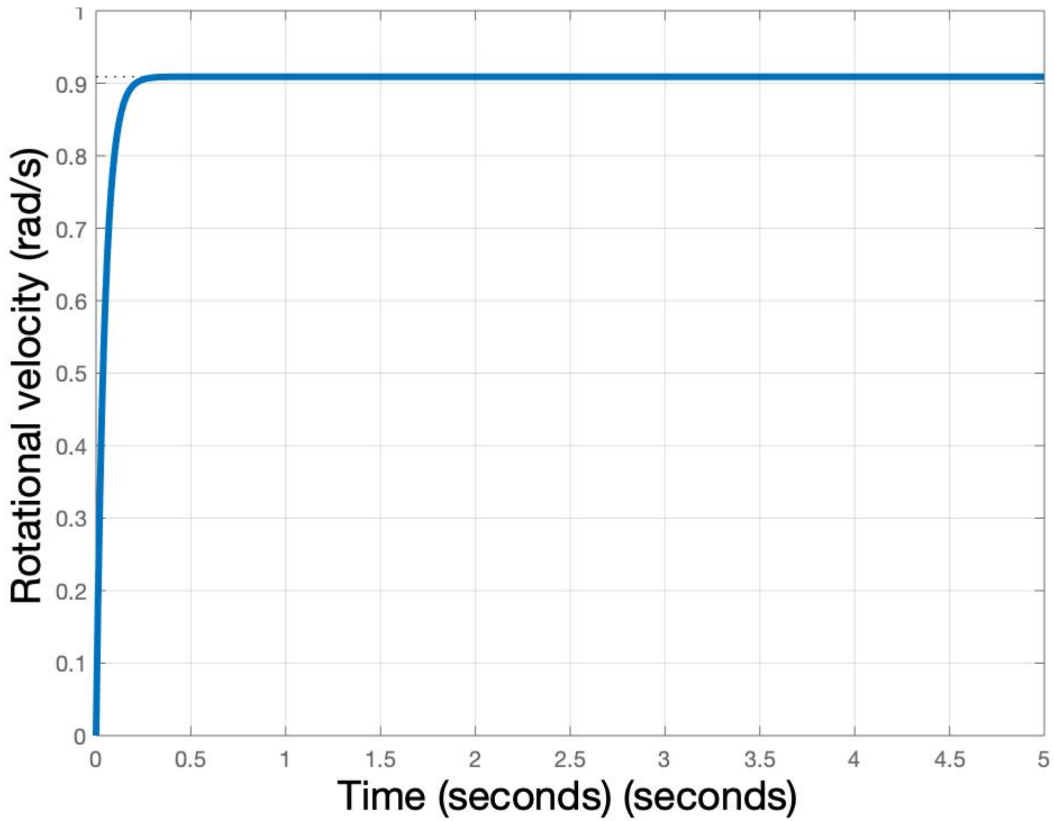
- Compute error signal $\mathbf{e} = \mathbf{r} - \mathbf{y}$
- Integral term: $K_I \int_0^t \mathbf{e}(\tau)d\tau$
  - K_I integral gain;
  - Feedback action proportional to cumulative error over time
  - If a small error persists, it will add up over time and push the system towards eliminating this error): eliminates offset/steady-state error



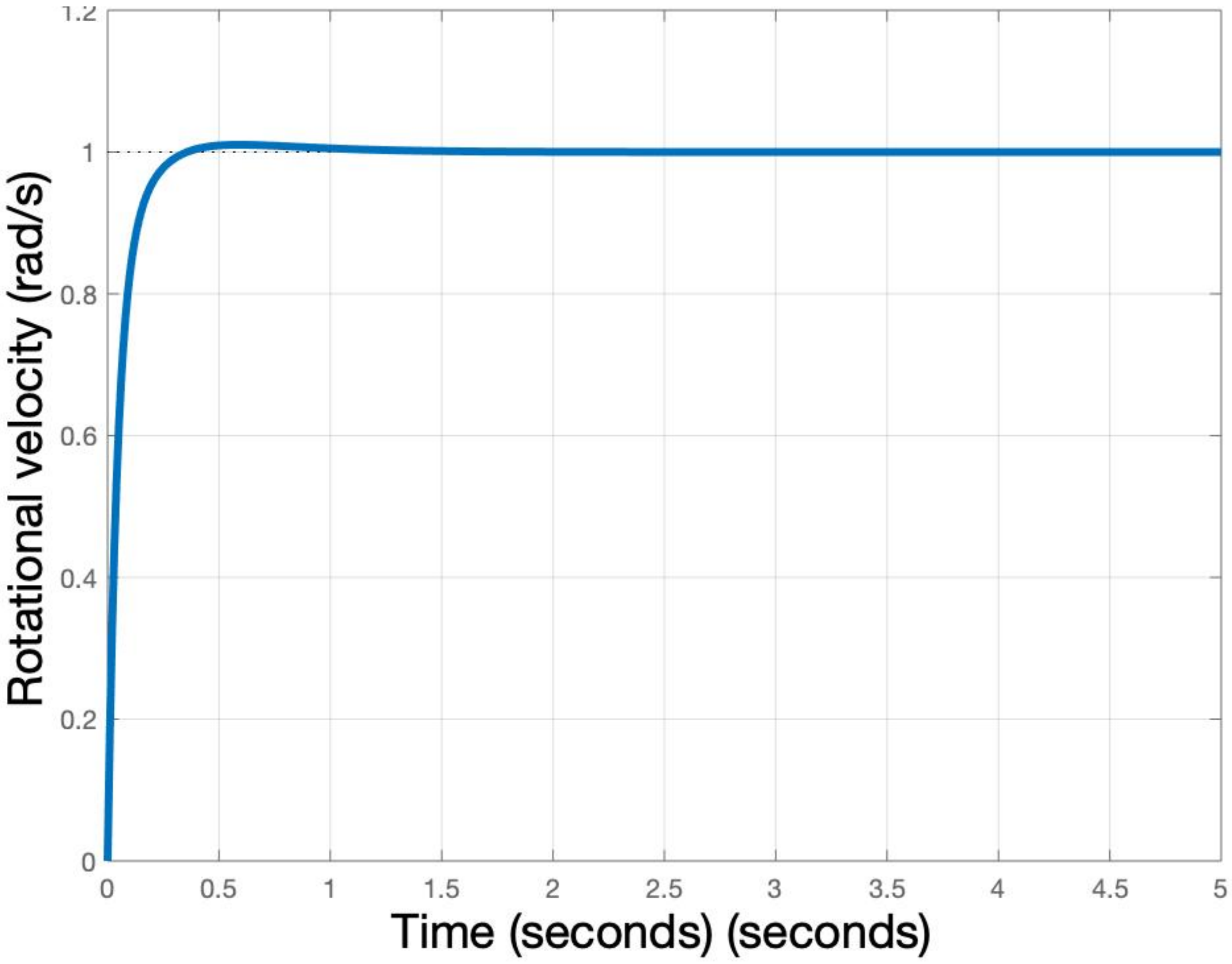
- Disadvantages:
  - Integral action by itself can increase instability
  - Integrator term can accumulate error and suggest corrections that are not feasible for the actuators (integrator windup)
    - Real systems “saturate” the integrator beyond a certain value

PD-controller

- Compute error signal $\mathbf{e} = \mathbf{r} - \mathbf{y}$
- Derivative term $K_d\dot{\mathbf{e}}$:
  - K_d derivative gain;
  - Feedback proportional to how fast the error is increasing/decreasing
- Purpose:
  - "Predictive" term, can reduce overshoot: if error is decreasing slowly, feedback is slower
  - Can improve tolerance to disturbances
- Disadvantages:
  - Still cannot eliminate steady-state error
  - High frequency disturbances can get amplified



PID-controller

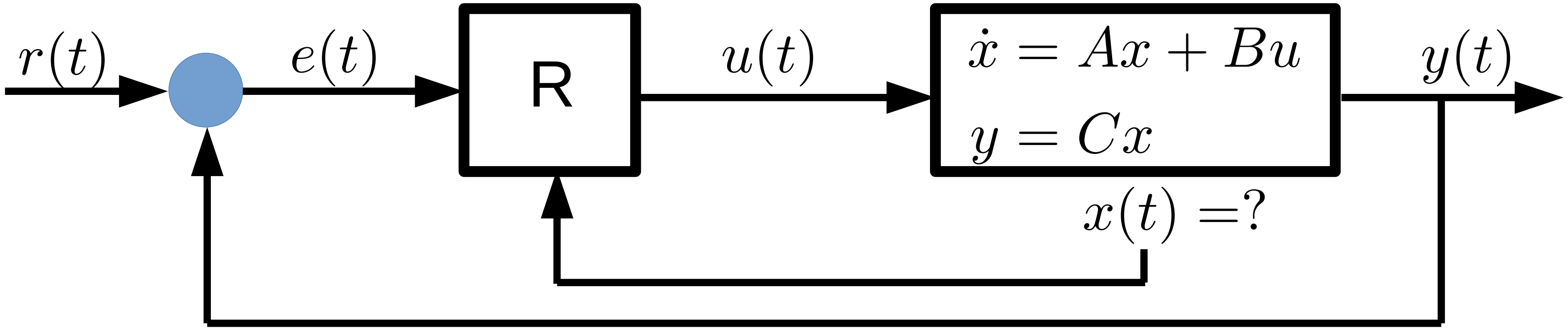


PID controller in practice

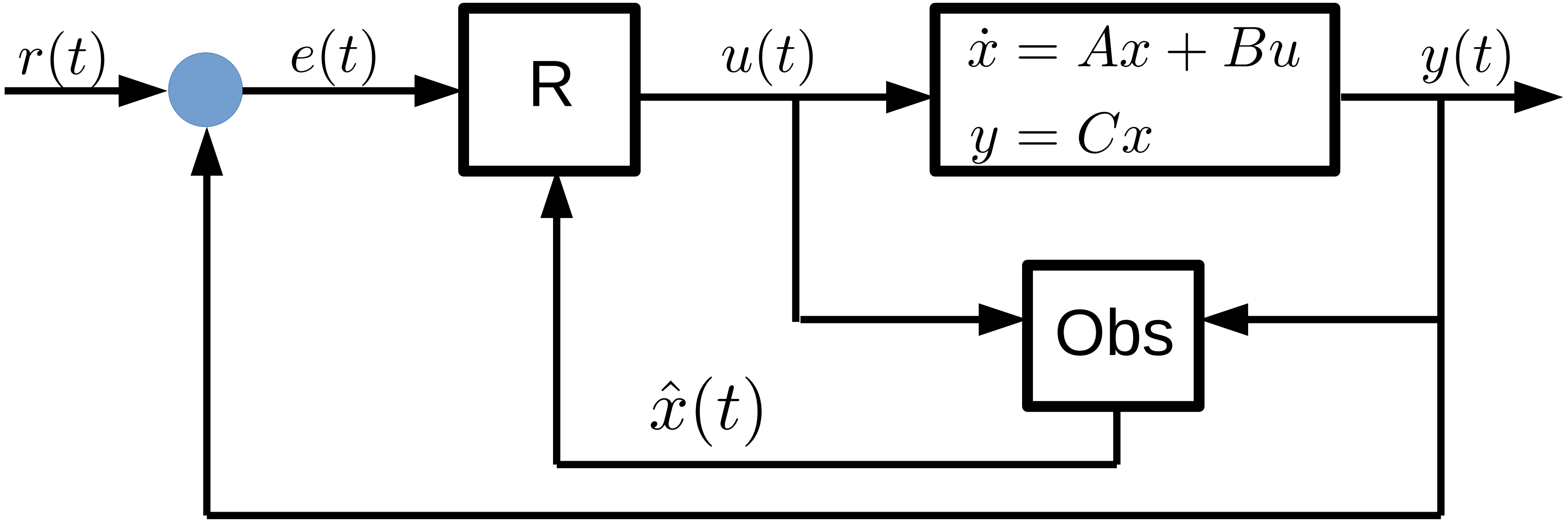
- May often use only PI or PD control
- Many heuristics to *tune* PID controllers, i.e., find values of K_P, K_I, K_D
- Several *recipes* to tune, usually rely on designer expertise
- E.g. *Ziegler-Nichols* method: increase K_P till system starts oscillating with period T (say till $K_P = K^*$), then set $K_P = 0.6K^*, K_I = \frac{1.2K^*}{T}, K_D = \frac{3}{40}K^*T$
- Matlab/Simulink has PID controller blocks + PID auto-tuning capabilities
- Work well with linear systems or for small perturbations,
- For non-linear systems use “gain-scheduling”
  - (i.e. using different K_P, K_I, K_D gains in different operating regimes)

Observation

- Problem:
  - Control design with (partially) unknown state



- Solution:
  - Luenberger Observer



Luenberger Observer

- State-space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$
$$\hat{y} = C\hat{x}$$
$$u = K(x_{ref} - \hat{x})$$

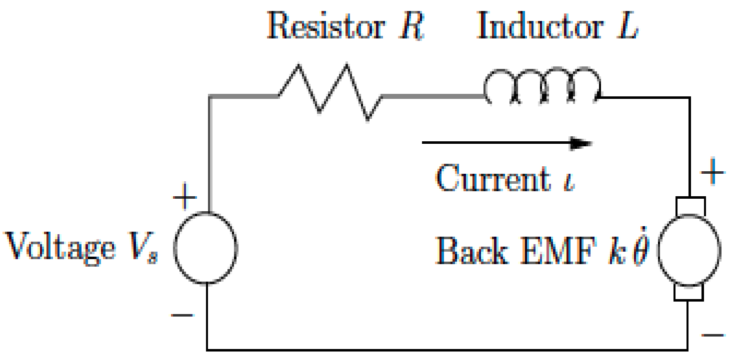
Control design parameters

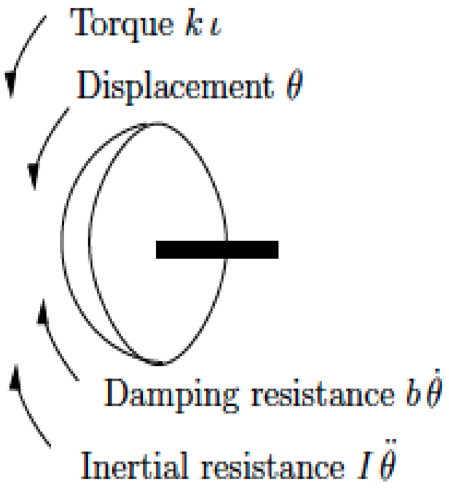
- Observer Error satisfies: $\dot{e} = (A - LC)e$
- Required: Observability, Controllability
- Pole Placement

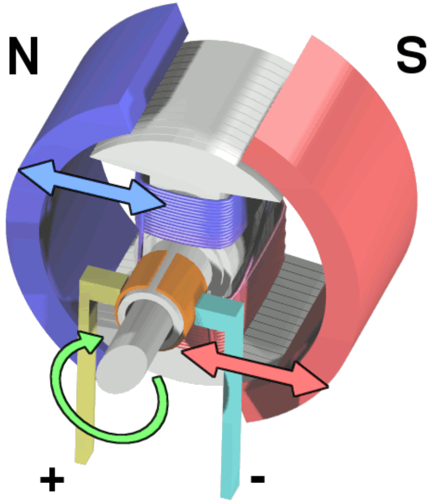
$$K : eig(A - BK) = \{\lambda_{c1}, \ldots, \lambda_{cn}\}$$
$$L : eig(A^T - LC) = \{\lambda_{o1}, \ldots, \lambda_{on}\}$$

Overall system is stable iff both observer and controller are stable

Example - DC Motor







b = 0.1 # friction coefficient (Nm/(rad/sec))
I = 0.01 # mechanical inertia (Kg*m^2)
k = 0.01 # motor torque constant (Nm/A)
R = 1   # armature resistance (Ohm)
L = 0.5 # armature inductance (H)

$$V_s = Ri + L\frac{di(t)}{dt} + k\theta_v$$
$$I\frac{d\theta_v}{dt} + b\theta_v = ki$$

State-space representation

$$\dot{x} = Ax + Bu$$

$$A = \begin{bmatrix} -b/I & k \\ -k/L & -R \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x = \begin{bmatrix} \theta_v \\ i \end{bmatrix} \quad u = V_s$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$