$$P = e^{-nh\nu/k_BT}\left(1 - e^{-h\nu/k_BT}\right) \quad (111)$$

SIMILARLY THE AVERAGE ENERGY OF SUCH A COLLECTION OF SYSTEMS IS

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} nh\nu\, e^{-nh\nu/k_BT}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_BT}} \quad (112)$$

THESE SUMS CAN BE WORKED OUT WITH A LOT OF ALGEBRA NOT REPORTED HERE TO GIVE

$$\langle E \rangle = \frac{h\nu}{e^{h\nu/k_BT} - 1} \quad (113)$$

FROM THIS AVERAGE ENERGY FORMULA, PLANCK WAS ABLE TO SHOW THAT THE E.M. ENERGY DENSITY $u_T(\nu)$ WITHIN THE B.B. CAVITY AT T IS

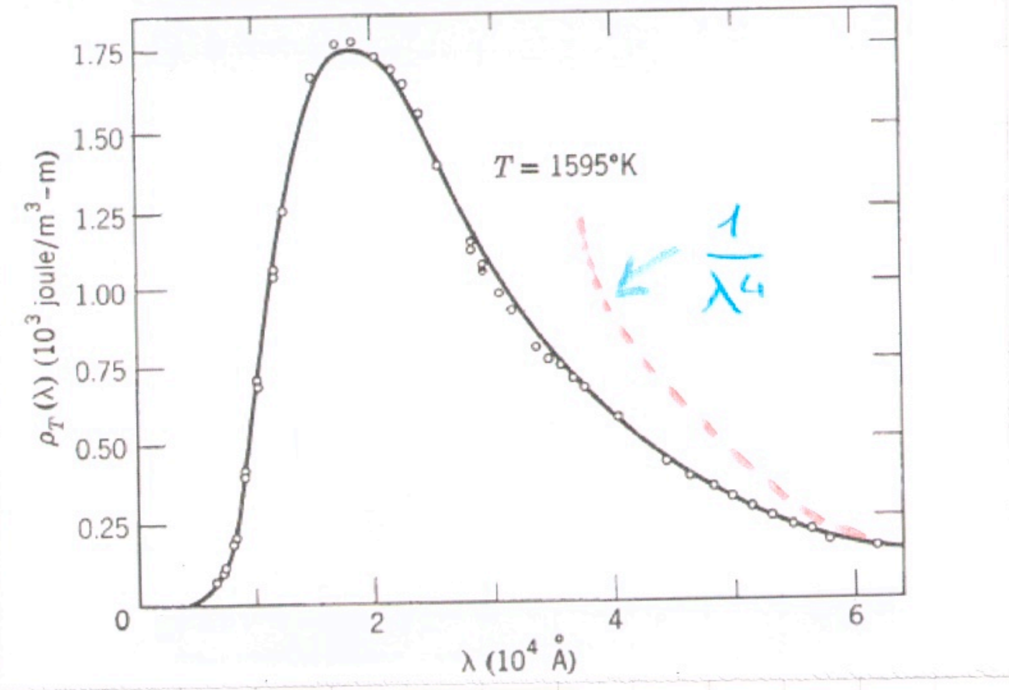
$$u_T(\nu)\,d\nu = \frac{8\pi h\nu^3}{c^3}\,\frac{1}{e^{h\nu/k_BT} - 1}\,d\nu \quad (114)$$

OR USING $\nu = c/\lambda$

$$\underbrace{u_T(\lambda)\,d\lambda}_{\text{E.M. ENERGY DENSITY}} = \frac{8\pi h}{\lambda^3}\,\frac{1}{e^{hc/\lambda k_BT} - 1}\,d\lambda \quad (115)$$

E.M. ENERGY DENSITY

IN THE INTERVAL $\lambda + d\lambda$. BY INTEGRATING WE OBTAIN $B_\lambda = \frac{2k_BT}{\lambda^4}$



OBSERVATION

PLANCK HAS INTRODUCED THREE NEW CONCEPTS.

1. THE ENERGY OF A PHYSICAL SYSTEM CAN TAKE ONLY DISCRETE VALUES. THE CLASSICAL ENERGY AND THE QUANTIZED ENERGY ARE INDISTINGUISHABLE ONLY IN THE LIMIT $K_B T > h\nu = E_{n+1} - E_n$ (110)

2. THE ENERGY EXCHANGED BETWEEN AN ABRAHAM-LORENTZ OSCILLATOR AND AE E.M FIELD IS QUANTIZED

3. THE E.M. ENERGY DENSITY IN A B.B. CAVITY IS RULED BY A STATISTICAL LAW AN IS NOT DETERMINISTIC.

PLANCK FORMULA RIVISITED

THE PLANCK EQUATION CAN BE OBTAINED ALSO BY ASSUMING THAT THE E.M. FIELD IS QUANTIZED. THE DETAILS OF THE QUANTIZATION OF THE E.M. FIELD WILL BE DISCUSSED LATER IN THIS LECTURES. AT THE MOMENT WE CAN MAKE THE ASSUMPTION THAT THE E.M. ENERGY OF AN E.M. MODE AT FREQUENCY ν IS DISCRETE AND NOT CONTINUUM AND THE QUANTUM OF THIS ENERGY IS $h\nu = \hbar\omega$.

THIS QUATUM OF E.M. ENERGY CAN BE SEEN AS A MASS-LESS PARTICLE, AS SUPPORTED BY THE SPECIAL RELATIVITY

$$E^2-(cp)^2 = m^2c^4 \Rightarrow p = E/c = h\nu/c$$

IT HAS SPIN = 1 IN UNITS OF $\hbar \Rightarrow$ IS A BOSON, WHILE THE LINEARITY OF THE MAXWELL EQS. IMPLIES THAT PHOTONS DO NOT INTERACT EACH OTHER. THE ENSAMBLE OF SUCH A PARTICLE CAN BE SEEN AS A BOSON GAS $\Rightarrow$ IT CAN BE TREATED IN THE FRAME OF THE BOSE-EINSTEIN (BE) STATISTICS. TO DEFINE THE EQUILIBRIUM CONDITION OF SUCH A GAS IT MUST INTERACT WITH MATTER. THE PLANCK EQUATION CAN BE DERIVED BY TREATING THE IDEAL PHOTON GAS AT EQUILIBRIUM BY MEAN

OF THE BE STATISTIC, WHICH IS OF COURSE A QUANTUM STATISTIC. BY STARTING FROM THE CLASSICAL BOLTZMANN STATISTIC THE PROBABLY DISTRIBUTION IS GIVEN BY

$$P(E) \propto \exp\left(-E/k_B T\right) \Rightarrow \qquad (117)$$

AN AVERAGE ENERGY PER MODE

$$\langle E \rangle = \frac{\int_0^\infty E\, P(E)\, dE}{\int_0^\infty P(E)\, dE} = k_B T \qquad (118)$$

IF WE POSTULATE $E = nh\nu$ (n = NUMBER OF PHOTONS IN THE MODE) $\Rightarrow$

$$P(E) = P(nh\nu) \propto \exp\left(\frac{-nh\nu}{k_B T}\right) \qquad (119)$$

AND THE $\langle E \rangle$ IS CALCULATED BY SUMMING ONLY OVER THE DISCRET ENERGY PERMITTED RATHER THAN INTEGRATING

$$\langle E \rangle = \frac{\sum'^{\infty}_{n=0} nh\nu\, P(nh\nu)}{\sum'^{\infty}_{n=0} P(nh\nu)} = \frac{\sum'^{\infty}_{n=0} nh\nu\, e^{\left(-\frac{nh\nu}{k_B T}\right)}}{\sum'^{\infty}_{n=0} e^{\left(-\frac{nh\nu}{k_B T}\right)}} \qquad (120)$$

THAT AFTER NON TRIVIAL CALCULATIONS GIVES

$$\langle E \rangle = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} \qquad (121)$$

WHICH IS THE SAME AS (113) LEADING TO (114) THAT INTEGRATED GIVES

THE PLANCK LAW

(122) $$B(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}$$

BY INTEGRATING THIS FUNCTION A T=COST

$$B_T(\nu) \equiv \int_0^\infty B_T(\nu)\, d\nu = \frac{\sigma T^4}{\pi} \quad (123)$$

WHERE $\sigma = \frac{2\pi^5 k_B^4}{15 c^2 h^3} \approx 5{,}67 \times 10^{5} \frac{erg}{cm^2 s K_B^4 sr}$

=> STEFAN-BOLTZMANN CONST.

THE ENERGY AT WHICH $B_T(\nu)$ IS MAX IS THE SOLUTION OF $\partial_\nu B_T(\nu) = 0$ AND

THIS GIVES $\nu_{MAX} \simeq \frac{59(GHz)}{T(K)}$ OR (124)

$\lambda_{MAX} \simeq \frac{0.29(cm)}{T(K)}$ THAT ARE THE WIEN'S

LAW.