$X \neq \emptyset$ irreducible : $X = X_1 \cup X_2$, $X_1, X_2$ closed $\Longrightarrow X_1 = X$ or $X_2 = X$

$U, V \subseteq X$ open, $\neq \emptyset \Longrightarrow U \cap V \neq \emptyset$

$X \subseteq \mathbb{A}^n, \mathbb{P}^n$ alg. variety, top. space w. Zariski closed

$X$ is irred. $\Longleftrightarrow$ $I(X)$ is prime, or $I_a(X)$ prime

$X$ top. space: $X$ is noetherian if:

1) ACC for open subsets

$$U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq \cdots \text{ open}$$

$$\Longrightarrow \exists\, r \text{ s.t. } U_r = U_{r+1} = U_{r+2} = \cdots$$

$\boxed{2)}$ DCC for closed subsets

3) condition of the maximal for non-empty sets of open subsets

$\boxed{4)}$ " " minimal " "

" closed subsets.

Ex: $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$ are noetherian w.r.t. Zariski topology

It follows from $K[x_1, \dots, x_n]$ is noetherian, $K$ field.

Take a descending chain of closed sets in $\mathbb{A}^n$:

$$X_1 \supseteq X_2 \supseteq \dots \supseteq X_r \supseteq \dots$$

$$I(X_1) \subseteq I(X_2) \subseteq \dots \subseteq I(X_r) \subseteq \dots$$

ascending chain of ideals in $K[x_1 \dots x_n]$ noeth.

$\Rightarrow$ stationary: $\exists\, r$ $I(X_r) = I(X_{r+i})$ $i \geq 0$

$$\begin{array}{ccccccc} V(I(X_1)) & \supseteq & V(I(X_2)) & \supseteq & \dots & \supseteq & \\ \| & & \| & & & & \\ X_1 & \supseteq & X_2 & \supseteq & \dots & X_r = X_{r+i} = \dots \end{array}$$

stationary

Proj. sp. $I_a(X_1) \subseteq I_a(X_2) \subseteq \dots$ stat.

$$\begin{array}{ccccc} V_P(I_a(X_1)) & \supseteq & V_P(I_a(X_2)) & \supseteq & \dots \\ \| & & \| & & \\ X_1 & & X_2 & & \dots \end{array}$$

**Prop.** $X$ noetherian topological space
$\emptyset \neq Y \subseteq X$ closed subset

Then: 1) $Y$ can be written $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r$ $(*)$ finite union
irreducible closed in $X$ and in $Y$

2) if $Y_i \not\subseteq Y_j$ $\forall i \neq j$, $Y_1, \ldots, Y_r$ are uniquely determined by $Y$: irreducible components of $Y$

**Pf.** By contradiction using $\boxed{4)}$ of def. of noeth. top. sp.
Ass. by contrad. that $\exists$ non-empty closed subsets in $X$ not finite union of irred. closed subsets.

$S = \{ Y \subseteq X \mid Y \neq \emptyset, \text{closed, not written as in } (*) \}$
$\neq \emptyset$ $\Longrightarrow$ $S$ has minimal elem.

$Z \in S$ minimal in $S \Rightarrow Z$ is not irred.

$Z = Z_1 \cup Z_2$, $Z_1, Z_2$ closed in $Z$, $Z_1 \subsetneq Z$, $Z_2 \subsetneq Z$, $Z_1 \neq \emptyset$, $Z_2 \neq \emptyset$

$Z$ minimal in $S \Rightarrow Z_1, Z_2 \notin S$,
$Z_1, Z_2$ can be written as finite union of irred. closed subsets $\Rightarrow Z = Z_1 \cup Z_2 \ldots$ contrad.

2) $Y = Y_1 \cup Y_2 \cup \dots \cup Y_r$ , $Y_1, \dots, Y_r$ closed irred.

If $\underbrace{Y_i}_{} \subseteq Y_j$ , we can eliminate $Y_i$ : from any expres. we can get a non-irredundant expression.

$Y = Y_1 \cup Y_2 \cup \dots \cup Y_r$ $\quad Y_i \not\subseteq Y_j \quad \forall i \neq j$

$= \boxed{Y_1'} \cup Y_2' \cup \dots \cup Y_s'$ $\quad , Y_1', \dots, Y_s'$ closed irred, $\quad Y_i' \not\subseteq Y_j'$

$Y_1' \subseteq Y_1 \cup Y_2 \cup \dots \cup Y_r \implies \exists\, j_1$ s.t. $Y_1' \subseteq Y_{j_1}$

irred.

$Y_{j_1} \subseteq Y_1' \cup Y_2' \cup \dots Y_s'$ : $\exists\, i_1$ s.t. $\underline{\underline{Y_1'}} \subseteq Y_{j_1} \subseteq \underline{\underline{Y_{i_1}'}} \implies$

$1 = i_1$ and $\boxed{Y_1' = Y_{j_1}}$

This can be repeated $\forall Y_i, Y_j' \implies$

the expressions are the same.

<u>Consequence</u> $\mathbb{A}^n_K, \mathbb{P}^n_K$ noeth. $\Longrightarrow$ if $X$ is an affine or a proj. algebraic variety

$X = X_1 \cup X_2 \cup \cdots \cup X_r$ <u>unique</u> finite union of irreducible alg. varieties, s.t. $X_i \not\subseteq X_j$ $\forall i \neq j$

$X_1, X_2, \dots, X_r$ are the irreducible components of $X$:

they are the maximal irreducible closed subsets of $X$

$X = X_1 \cup \dots \cup X_r \supseteq Y$ irred. closed $\Longrightarrow \exists\, i \in \{1, \dots, r\}$ $Y \subseteq X_i$ so $X_1, \dots, X_r$ are maximal among irred. closed subsets.

If $K$ is algebraically closed:

$\{$ Irred. closed subsets of $\mathbb{A}^n_K \} \longleftrightarrow \{$ prime ideals of $K[x_1 \dots x_n] \}$

$\{$ " " " $\subseteq X \} \longleftrightarrow \{$ prime ideals of $K[x_1 \dots x_n]$ containing $I(X) \}$

$X$ closed

$\{$ maximal irred. " " $\subseteq X \} \longleftrightarrow \{$ minimal prime ideals containing $I(X) \}$

"

$\{$ irred. components of $X$

Often algebraic $\boxed{\text{variety}}$ means irreducible alg. set

More general notion: quasi-projective variety

$X$ top. space: locally closed subset of $X$ is

$\boxed{C \cap U}$, $C$ closed, $U$ open

In partic.: $C$ closed $\quad C = C \cap X \quad$ loc. closed

$U$ open $\quad U = U \cap X \quad$ " "

$Z$ is locally closed means $Z$ is open in a closed or closed in an open subset

$X = \mathbb{P}^n_K \quad Z$ locally closed $\quad Z = Y \cap U$

closed $\quad$ open

In partic. $U = U_0 = \mathbb{A}^n_K$

$Y \cap U_0$ : all affine alg. sets are locally closed in $\mathbb{P}^n_K$

Def. a quasi-projective variety is an irreducible locally closed subset of a $\mathbb{P}^n_K$

$Z = Y \cap U \qquad Z$ irred., open in $Y$

cl. $\quad$ open

$\overline{Z}$ closed irred. $\qquad Z = \overline{Z} \cap U$: it is the closure of $Z$ in $U$ but $Z$ is already closed in $U$

$Z = \overline{Z} \cap U, \quad \overline{Z}$ irred.

$Z$ is dense in $\overline{Z}$

|| A quasi-proj. var. is a dense open subset of an irred. proj. variety.

**Prop.** $X \subseteq \mathbb{A}^n_K$, $Y \subseteq \mathbb{A}^m_K$ irreducible affine var.
Then $X \times Y \subseteq \mathbb{A}^{n+m}_K = \mathbb{A}^n_K \times \mathbb{A}^m_K$ is irreducible

**Pf.** $X \times Y = W_1 \cup W_2$, $W_1, W_2$ closed: the claim is that either $W_1 = X \times Y$ or $W_2 = X \times Y$.

$P \in X$ $(a_1 \dots a_n)$ $\quad \underbrace{\{P\} \times Y}_{\text{closed}} \subseteq X \times Y$

$\{P\} \times Y$ is homeomorphic to $Y$:

$\{P\} \times Y = \{(a_1 \dots, a_n, y_1 \dots, y_m) \mid (y_1 \dots y_m) \in Y\}$

$\{P\} \times Y \subseteq \mathbb{A}^{n+m} \xrightarrow{\varphi} Y \subseteq \mathbb{A}^m$

$(a_1 \dots a_n, y_1 \dots y_m) \longrightarrow (y_1 \dots y_m)$

$Z \subseteq Y$ closed $\quad \varphi^{-1}(Z) = \{(a_1 \dots a_n, y_1 \dots y_m) \mid F_i(y_1 \dots y_m) = 0 \ \forall i\}$

$Z = V(F_1 \dots, F_r)$ $\quad \varphi^{-1}(Z) = V(F_1 \dots F_r) \cap (\{P\} \times \mathbb{A}^m)$ closed

Similarly for the map $\varphi^{-1}$.

$Y$ irred. $\Longrightarrow$ $\boxed{\forall P \in X}$ $\{P\} \times Y$ irred.

$\{P\} \times Y = (\underbrace{W_1 \cap (\{P\} \times Y)}_{\text{closed}}) \cup (\underbrace{W_2 \cap (\{P\} \times Y)}_{\text{closed}}) \Longrightarrow$

$W_1 \cup W_2 = X \times Y$

$\exists\, i \in \{1,2\}$ s.t. $\{P\} \times Y = W_i \cap (\{P\} \times Y) \Longrightarrow$

$W_i \supseteq \underline{\{P\} \times Y}$ Def. $X_i = \{P \in X \mid \{P\} \times Y \subseteq W_i\}$

$X = X_1 \cup X_2$ : $\boxed{\underline{\text{Claim}}\ X_1, X_2 \text{ are closed in } X}$

Assuming the claim, $X$ irred. $\Longrightarrow$ $X = X_1$ or $X = X_2$ ;

if $X = X_1$: $\Rightarrow$ $\forall P \in X$ $\{P\} \times Y \subseteq W_1$. $\Longrightarrow$ $X \times Y \subseteq W_1$.

<u>Pf of claim</u> that $X_i$ is closed in $X$

$Q \in Y$ def. $X^i(Q) \subseteq X$ $X^i(Q) = \{P \in X \mid (P,Q) \in W_i\} \subseteq X$

$X \times \{Q\}$ homeom. to $X$ and closed

$\underbrace{(X \times \{Q\}) \cap W_i}_{\text{closed in } X \times Y} = X^i(Q) \times \{Q\}$ homeom. to $X^i(Q)$

$\Longrightarrow$ $X^i(Q)$ is closed in $X$

$X_i = \bigcap_{Q \in Y} \underbrace{X^i(Q)}_{\text{closed in } X}$ $\Longrightarrow$ it is closed.

$X$ topological space

**Def.** topological dimension or combinatorial dim of $X$

$\dim X$ is the sup. of the lengths of the finite chains of irreducible closed subsets of $X$:

$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$ closed irreducible in $X$

length of this chain is $n$ = number of inclusions

$\dim X$ — $\infty$
— natural number $n$

**Ex.** if $x \in X$ and $x$ is closed $\Rightarrow \{x\}$

Among the chains to be considered

$\{x\} \subsetneq \cdots$

$\mathbb{R}^n \ni x$ : if $X \subseteq \mathbb{R}^n$ closed in the eucl. top

$X \supsetneq \{x\} \Rightarrow X$ reducible

$\mathbb{A}^1_K$ , $K$ infinite , $\{P\} \subsetneq \mathbb{A}^1$ only chain of length $> 0$ containing $P$

chains of length 0 : $\{P\}$
$\mathbb{A}^1$

chains of lg 1 $\{P\} \subsetneq \mathbb{A}^1$ $\Longrightarrow$ $\dim \mathbb{A}^1_K = 1$

$\mathbb{A}^n_K$ : there exist chains of length $n$

In $K[x_1, \dots, x_n]$: $(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \dots \subsetneq (x_1, \dots, x_n) \subsetneq K[x_1 \dots x_n]$

$$\underset{\text{irred.}}{V(0) = \mathbb{A}^n} \supsetneq \underset{\substack{\| \\ H_1 \\ \text{hyperplane}}}{V(x_1)} \supsetneq V(x_1, x_2) \supsetneq \dots \supsetneq \underset{\substack{\| \\ \{0\}}}{V(x_1, \dots, x_n)}$$

Each $V(x_1, \dots, x_i)$ is a linear subspace of $\mathbb{A}^n$, it is itself an affine space of dim $n-i$ : irred.

$\Longrightarrow$ chain of length $n$

$\dim \mathbb{A}^n_K \geq n$ : to prove that $=$ , need to prove that there are no longer chains.

$X$ irred. top. space

$\dim X = 0 \implies$ all chains of irred. closed subsets have length $0$

$\implies$ the only closed irred subset of $X$ is $X$

$x \in X$ $\qquad \{x\}$ is irred. even if not nec. closed

$\implies \overline{\{x\}}$ is irred. $\implies \boxed{\overline{\{x\}} = X}$