X topological space

$\dim X = \sup \{ \text{lengths of chains}$

$$\underbrace{X_0 \subset X_1 \subset \cdots \cdots \subset X_r}_{r} \quad \text{irreducible closed in } X \Big\}$$

$\dim \mathbb{A}^n_K \leq n$

<u>Proposition</u> 1) X top. sp., $Y \subseteq X$: $\boxed{\dim Y \leq \dim X}$

2) $X = \bigcup_{i \in I} U_i$ open covering : $\dim X = \sup_{i \in I} \{ \dim U_i \}$

3) X noetherian, $X = \underbrace{X_1 \cup \cdots \cup X_r}_{\text{irred. components}} \Rightarrow \dim X = \sup_{i=1,\dots,r} \{ \dim X_i \}$

4) Ass. $Y \subseteq X$ closed, $\dim X$ is finite, X is irred.
If $\dim X = \dim Y \Rightarrow X = Y$

1) $Y \subseteq X$

$Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_r$ irred. closed subsets of Y

$\overline{Y_0} \subsetneq \overline{Y_1} \subsetneq \quad \subsetneq \overline{Y_r}$ closed in X, irreducible

$\overline{Y_i} \cap Y$ it is the closure of Y_i in Y

$\overset{\shortparallel}{Y_i}$ because Y_i is closed in Y

$\Rightarrow \overline{Y_0} \subsetneq \overline{Y_1} \subsetneq \cdots \subsetneq \overline{Y_r}$

From a chain of length r in $Y \Rightarrow$ chain of length r in X

$\Rightarrow \dim Y \leq \dim X$.

2) $X = \bigcup_{i \in I} U_i$

$\emptyset \neq \underline{\underline{X_0}} \subsetneq X_1 \subsetneq \cdots \subsetneq X_r$ irred. cl. in X

$\exists\, i$ s.t. $\underset{\neq \emptyset}{X_0 \cap U_i} \subseteq X_1 \cap U_i \subseteq \cdots \subseteq X_r \cap U_i$ closed in U_i irred.

$\forall j = 0, \ldots, r$ $X_j \cap U_i$ is open in X_j, irred. $\Rightarrow$ irred.

$\overline{X_j \cap U_i} = X_j$ because X_j is irred. and $X_j \cap U_i$ is open dense

$\Rightarrow$ inclusion are strict

$\Rightarrow \dim X \leq \sup_i \dim U_i$

$\dim U_i \leq \dim X \quad \forall i$

$\sup \dim U_i \leq \dim X$

3) $X = X_1 \cup \dots \cup X_r$
irred. comp.

$Z_0 \subset Z_1 \subset \dots \subset \underbrace{Z_n}$ irred. closed subsets in X

$Z_n \subseteq X = X_1 \cup \dots \cup X_r \Longrightarrow Z_n \subseteq X_i$.

$\Downarrow$

this is a chain of irred. cl. subsets of X_i, for some X_i.

$\dim X \leq \sup_i \dim X_i$, - as in 2)

4) $Y \subseteq X$ closed, X irred., $\dim X$ finite, $\dim Y = \dim X$

$Y_0 \subset Y_1 \subset \dots \subset Y_n \subseteq X$ irred. cl. in $Y \Longrightarrow$

also closed in X, X irred.

$\Downarrow$

$Y_n = X \subseteq Y$

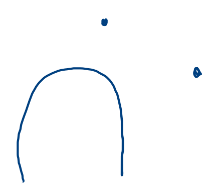
$n = \dim X = \dim Y$

$\Longrightarrow \boxed{X = Y}$

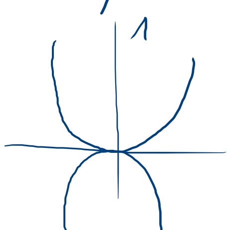
a) $X = V(y + x^2)$ homeom. to $\mathbb{A}^1$: $\dim X = 1$

$X \cup \{A\} \cup \{B\}$ $\quad \dim A = 0 = \dim B$

1 $\quad$ points $\quad$ 0



b) $\underbrace{V(y + x^2)}_{1} \cup \underbrace{V(y - x^2)}_{1} = \underbrace{X}_{1}$, $\quad Y = V(y - x^2)$



$Y \subsetneq X \quad \dim Y = \dim X$

$Y \subseteq X$ $\quad \dim Y \leq \dim X$ $\quad \dim X$ finite

codimension of Y in X $= \dim X - \dim Y$

__Cor.__ K infinite

$\dim A^n_K = \dim \mathbb{P}^n_K$

$\mathbb{P}^n_K = U_0 \cup U_1 \cup \dots \cup U_n$ $\qquad U_i = \mathbb{P}^n \setminus V_p(x_i)$

U_i is homeom. to A^n $\therefore$ $\dim U_i = \underline{\dim A^n}$

$\Longrightarrow \dim \mathbb{P}^n = \dim A^n$

$U_0 \subsetneq \mathbb{P}^n$ $\qquad \dim U_0 = \dim \mathbb{P}^n$

same dim but strict inclusion

X noeth. $\qquad X = X_1 \cup \dots \cup X_r$ irred. comp.

If $\dim X_1 = \dots = \dim X_r = \dim X$

__Def.__ X has __pure__ dimension $d \iff$

all its irred. comp. have dim d

$\mathbb{A}^n_K \supset X$ affine alg. set

<u>Def</u>. the coordinate ring of X is $\dfrac{K[x_1, \dots, x_n]}{I(X)} = K[X]$

$K[X]$ is a finitely generated K-algebra
$\| $
$K[t_1, \dots, t_n] \qquad t_i = [x_i]$

$K[X]$ can be interpreted the ring of polynomial functions on X

$\varphi: K[x_1, \dots, x_n] \longrightarrow \mathcal{F}(X) = \{ f : X \to K \}$ functions on X

$F \longrightarrow f$

is a ring with pointwise defined operations

f is the polynom. function on X s.t.

$f(P) = F(P)$

$f+g$ is the map such that

$(f+g)(P) = f(P) + g(P)$

φ is a ring homom.

$(fg)(P) = f(P)g(P)$

$\operatorname{Im} \varphi = \{$ polynomial functions on $X \}$

$\operatorname{Ker} \varphi = \{ F \in K[x_1 \dots x_n] \mid \varphi(F) = f = 0$ the zero function on $X \}$

$= I(X)$

$\operatorname{Im} \varphi \simeq \dfrac{K[x_1, \dots, x_n]}{I(X)} = \underline{K[X]}$

ring of polynomial functions on X

$t_i = [x_i]$

$\varphi : x_i \longrightarrow$ function $X \longrightarrow K$

$P(a_1 \dots a_n) \longrightarrow x_i(P) = a_i$

t_i

$t_1, \dots, t_n$: coordinate functions on X

$K[X] = K[t_1, \dots, t_n]$: it is the K-algebra generated by the coordinate on X

R ring : Krull dimension of R $\dim R$ is
sup of the lengths of chains of prime ideals of R
$P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_r$

If K is algebraically closed :

Prop The chains of prime ideals of $K[X]$ are in bijection with the chains of irred. closed subsets of X

If $K[X] = \frac{K[x_1 \dots x_n]}{I(X)}$

The ideals of $K[X]$ are of the form $\frac{\alpha}{I(X)}$ where
$\alpha \subseteq K[x_1 \dots x_n]$ ideal s.t. $\alpha \supseteq I(X)$

Prime ideals of $K[X]$ are $P/I(X)$, where
$P \supseteq I(X)$ is a prime ideal.

$\alpha/I(X)$ is prime $\Longleftrightarrow$ α is prime

Nullstellensatz :

The irred. components of X are the maximal irred. closed subsets of X $\longleftrightarrow$ minimal prime ideals in $K[x_1 \dots x_n]$ containing $I(X)$

Irred. closed subsets of $X \longleftrightarrow$ prime ideals containing $I(X)$
$\longleftrightarrow$ prime ideals of $\frac{K[x_1 \dots x_n]}{I(X)} = K[X]$

chains of irred. closed subsets of $X \longleftrightarrow$
" " prime ideals in $K[X]$

Take sup of lengths of chains $\Longrightarrow$

$$\boxed{\dim X = \dim K[X]}$$

$X_0 \subset X_1 \dots \subset X_r$ in X

$I(X_0) \supset \dots \supset I(X_r)$ prime ideals $\supseteq I(X)$

$\frac{I(X_0)}{I(X)} \supset \dots \supset \frac{I(X_r)}{I(X)}$ " " in $K[X]$

$V(I(X_0)) \supset V(I(X_1)) \supset \dots \supseteq V(I(X_r))$
$\quad = X_0 \qquad = X_1 \qquad\qquad = X_r$

the length is the same

$P_0 \subset P_1 \subset \dots \subset P_r$ prime ideals in $K[X]$
$P_0 = \frac{Q_0}{I(X)}$, $P_1 = \frac{Q_1}{I(X)}$, $P_r = \frac{Q_r}{I(X)}$

$Q_0 \subset Q_1 \subset \dots \subset Q_r$ prime ideals in $K[x_1 \dots x_n]$

$$\boxed{V(Q_0) \supseteq V(Q_1) \supseteq \dots \supseteq V(Q_r)}$$

$I(V(Q_0)) \subseteq I(V(Q_1)) \subseteq \dots \subseteq I(V(Q_r))$

If K is not alg. closed, we cannot conclude that
this is the chain $Q_0 \subset Q_1 \subset \dots \subseteq Q_r$

$\dim X \leq \dim K[X]$