Th. ii) let $u \in \mathcal{C}_0(\Omega)$
then $\rho_\varepsilon * u \xrightarrow{\varepsilon\to 0} u$ uniformly

iii) let $u \in L^p(\Omega)$ with $1 \le p < +\infty$
then $(\rho_\varepsilon * \bar{u})|_\Omega \xrightarrow{\varepsilon \to 0} u$ in $L^p(\Omega)$
where $\bar{u} = \begin{cases} u & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega \end{cases}$

proof. if $\varepsilon_0 = \text{dist}(\text{supp}\, u, \partial\Omega)$
if $0 < \varepsilon < \varepsilon_0$
then $\underbrace{\text{supp}\,(\rho_\varepsilon * u)}_{\substack{\cap \\ \text{supp}\,\rho_\varepsilon + \text{supp}\, u}} \subseteq \Omega$ and it is compact set

and $\rho_\varepsilon * u(x) = \int_{|y|\le\varepsilon} \rho_\varepsilon(y)\, u(x-y)\, dy$

so that $(\rho_\varepsilon * u - u)(x) = \int_{|y|\le\varepsilon} \rho_\varepsilon(y)\,(u(x-y) - u(x))\, dy$

$\left(\int_{|y|\le\varepsilon} \rho_\varepsilon(y)\, dy = 1 \right)$

$u \in \mathcal{C}_0(\Omega)$ (continuous with compact supp. in Ω) $\Rightarrow$ uniformly continuous

$\forall \tau > 0,\ \exists \delta > 0 : \forall x_1, x_2 \quad |x_1 - x_2| < \delta \Rightarrow |u(x_1) - u(x_2)| < \tau$

so that if $\varepsilon < \delta$ and $|y| \le \varepsilon$ then
$|u(x-y) - u(x)| < \tau$

so that, if $\varepsilon < \delta$: $|(\rho_\varepsilon * u - u)(x)| \le \int_{|y|\le\varepsilon} \rho_\varepsilon(y) \underbrace{|u(x-y) - u(x)|}_{<\tau}\, dy \le \tau \quad \forall x \in \Omega$

now I prove iii)
let $u \in L^p(\Omega)$ then $\bar{u} \in L^p(\mathbb{R}^d)$
fix $\delta > 0$ then $\exists w \in \mathcal{C}_0(\mathbb{R}^d)$ s.t.
$\|\bar{u} - w\|_{L^p(\mathbb{R}^d)} < \delta$ (density of $\mathcal{C}_0(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$)

consider $\|\underset{\in L^p}{\rho_\varepsilon * \bar{u}} - \underset{\in L^p}{\rho_\varepsilon * w}\|_{L^p(\mathbb{R}^d)}$
$= \|\rho_\varepsilon * (\bar{u} - w)\|_{L^p} \le \underbrace{\|\rho_\varepsilon\|_{L^1}}_{=1} \|\bar{u} - w\|_{L^p} < \delta$

so
$\|(\rho_\varepsilon * \bar{u})|_\Omega - u\|_{L^p(\Omega)} \le \|\rho_\varepsilon * \bar{u} - \bar{u}\|_{L^p(\mathbb{R}^d)} \le$

$\le \|\rho_\varepsilon * \bar{u} - \rho_\varepsilon * w + \rho_\varepsilon * w - w + w - \bar{u}\|_{L^p}$

$\le \underbrace{\|\rho_\varepsilon * \bar{u} - \rho_\varepsilon * w\|_{L^p}}_{<\delta} + \|\rho_\varepsilon * w - w\|_{L^p} + \underbrace{\|w - \bar{u}\|_{L^p}}_{<\delta}$

$w \in \mathcal{C}_0(\mathbb{R}^d)$ so $\rho_\varepsilon * w \to w$ uniformly in $\mathcal{C}_0(\mathbb{R}^d)$
$\Downarrow$
$\rho_\varepsilon * w \to w$ in $L^p(\mathbb{R}^d)$

so if ε is sufficiently small $\|\rho_\varepsilon * w - w\|_{L^p(\mathbb{R}^d)} < \delta$

conclusion $\forall \delta > 0,\ \exists \varepsilon_0 > 0 : 0 < \varepsilon < \varepsilon_0$

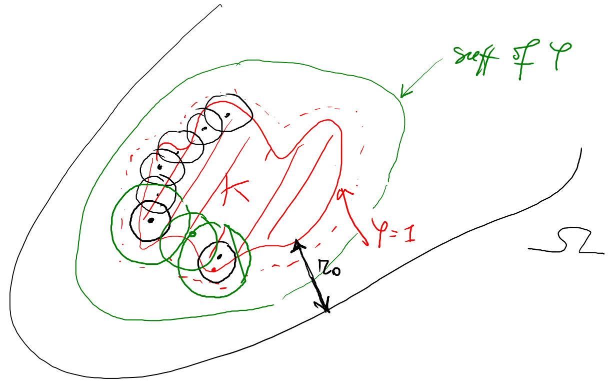
$\Rightarrow \|(\rho_\varepsilon * \bar{u})|_\Omega - u\|_{L^p(\Omega)} < 3\delta$

QED

Rem. mollifiers are usefull also to construct functions which value is 1 in a nbh of a compact K

C_0^∞

¿ $\exists\ \varphi \in C_0^\infty(\Omega)$ s.t. $\varphi(x) = 1$ $\forall x$ in an open set containing K



consider $r_0 = dist(K, \partial\Omega)$

take $\{x\in K, B(x, r_0/4)\} \longrightarrow$ open covering of K

extract a finite subcovering $B(x_1, r_0/4), \dots, B(x_N, r_0/4)$

take $\tilde K = \overline{\bigcup_{j=1}^{N} B(x_j, r_0/2)}$ $\qquad K \subseteq \tilde K$

take $\varphi(x) = \rho_{r_0/4} * \chi_{\tilde K}(x)$ $\qquad \chi_{\tilde K}(y) = \begin{cases} 1 & \text{if } y \in \tilde K \\ 0 & \text{if } y \notin \tilde K \end{cases}$

exercise : verify that $\varphi(x) = 1$ in $\bigcup_{j=1}^{N} B(x_j, r_0/4)$

if $x \in B(x_1, r/4)$

$$\rho_{r_0/4} * \chi_{\tilde K}(x) = \int_{|y| \le r_0/4} \rho_{r_0/4}(y) \underbrace{\chi_{\tilde K}(x-y)}_{=1} = 1$$

if $x \in B(x_1, r_0/4)$, $y \in B(0, r_0/4)$

rem. 2 Let $f \in L^1_{loc}(\Omega)$

Suppose $\forall \varphi \in C_0^\infty(\Omega)$, $\int_\Omega f\cdot\varphi = 0$ test with less functions

then $f = 0$

we know that if $\forall \psi \in C_0(\Omega)$, $\int_\Omega f\psi = 0$

then $f = 0$.

test with more functions

$C_0^\infty(\Omega) \subsetneq C_0(\Omega)$

take $\psi \in C_0(\Omega)$

take $(\rho_n)_n$ consider $\rho_n * \psi \in C_0^\infty(\Omega)$

$(\forall n \geq \bar{n})$

and $\rho_n * \psi \longrightarrow \psi$ uniformly ($\Rightarrow$ pointwise)

and $\operatorname{supp}(\rho_n * \psi) \subseteq \underset{\text{fixed compact}}{K} = \overline{(\operatorname{supp}\psi) + \operatorname{supp} B(0,1)}$

$$\left| f(\rho_n * \psi) \right| \leq \left| f \cdot \chi_K \right| \cdot \underbrace{\|\rho_n\|_{L^1} \, \|\psi\|_{L^\infty}}_{\text{constant.}}$$

L

Apply dominated convergence:

$$\underset{0}{\int_\Omega f(\rho_n * \psi)} \longrightarrow \int_\Omega f\psi$$

So that I proved

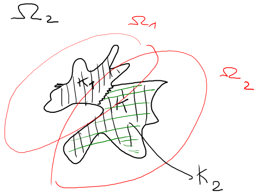
if $\psi \in C_0(\Omega)$ then $\int f\psi = 0 \Rightarrow f = 0$

Partition of Unity

Exercise. take K compact in $\mathbb{R}^d$ (also $\mathbb{R}^2$)

suppose Ω_1, Ω_2 are open sets in $\mathbb{R}^d$

s.t. $K \subseteq \Omega_1 \cup \Omega_2$



then $\exists K_1, K_2$ compact sets

$K_1 \subseteq \Omega_1$

$K_2 \subseteq \Omega_2$

s.t. $K = K_1 \cup K_2$

Theorem. Let K be a compact set

suppose $\Omega_1, \Omega_2, \dots \Omega_N$ open sets

such that $K \subseteq \bigcup_{j=1}^{N} \Omega_j$

then $\exists \varphi_1, \varphi_2, \dots \varphi_N$ s.t. $\forall j \quad \varphi_j \in \mathcal{C}_0^\infty(\Omega_j)$

s.t. $\forall x \in K, \quad \sum_{j=1}^{N} \varphi_j(x) = 1$

proof. using the the exercise we have

$K_1, K_2, \dots, K_N$ compact sets

s.t. $K_j \subseteq \Omega_j$ and $\bigcup_{j=1}^{N} K_j = K$

consider now $\psi_j \in \mathcal{C}_0^\infty(\Omega_j)$ s.t. $\psi_j = 1$ in a nbh. of K_j

Take
$$\varphi_1 = \psi_1$$
$$\varphi_2 = \psi_2(1-\psi_1)$$
$$\varphi_3 = \psi_3(1-\psi_2)(1-\psi_1)$$
$$\vdots$$
$$\varphi_N = \psi_N(1-\psi_{N-1})\cdots(1-\psi_1)$$

$\varphi_j \in \mathcal{C}_0^\infty(\Omega_j)$ (because $\psi_j \in \mathcal{C}_0^\infty(\Omega_j)$ and $(1-\psi_{j-1})\cdots(1-\psi_1) \in \mathcal{C}^\infty(\mathbb{R}^d)$)

and take $x \in K$

$\forall x \longrightarrow \sum_{j=1}^{N} \varphi_j(x) = 1 - (1-\psi_1)(1-\psi_2)\cdots(1-\psi_N)$ (prove by induction on N)

$x \in K \Rightarrow x \in K_j$ for some j

$\Rightarrow \psi_j(x) = 1$ so that $\sum \varphi_j = 1$

QED

Distributions

Notations $\alpha \in \mathbb{N}^n$

multi index $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n)$

lenght of the multi index $|\alpha| = (\alpha_1 + \alpha_2 + \dots + \alpha_n)$

if $x \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$, $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \dots \cdot x_n^{\alpha_n}$

if $\alpha \in \mathbb{N}^n$, $\partial_x^\alpha = \dfrac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}}$

$\left(f \in \mathcal{C}^\infty(\mathbb{R}^2) \qquad \partial_x^\alpha f = \dfrac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}} f \right)$

i = imaginary unit

$\left(\text{Hörmander's notation} \qquad D^\alpha = (-i)^{|\alpha|} \partial_x^\alpha \right)$

$\alpha \in \mathbb{N}^n \qquad \alpha! = \alpha_1! \alpha_2! \alpha_3! \dots \alpha_n!$

$\alpha, \beta \in \mathbb{N}^n$, $\alpha \le \beta$ will mean $\alpha_1 \le \beta_1, \dots, \alpha_n \le \beta_n$

and given $\alpha \le \beta$ $\qquad \dbinom{\beta}{\alpha} = \left(\dfrac{\beta_1!}{\alpha_1!(\beta_1 - \alpha_1)!} \cdot \dots \cdot \dfrac{\beta_n!}{\alpha_n!(\beta_n - \alpha_n)!} \right)$

def. Let $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$)
$\qquad\qquad\quad \parallel$
$\qquad\qquad (\mathcal{C}_0^\infty(\Omega))$

if i) T is linear $\Big(\forall a, b \in \mathbb{R}$ (or $\mathbb{C}$) $\forall \varphi, \psi \in \mathcal{D}(\Omega)$
$$T(a\varphi + b\psi) = aT(\varphi) + bT(\psi) \Big)$$

ii) $\forall K$ compact set in Ω, $\exists C_K > 0$, $m_K \in \mathbb{N}$,
s.t.
$$|T(\varphi)| \leq C_K \cdot \sum_{|\alpha| \leq m_K} \sup_{\Omega} |\partial_x^\alpha \varphi| \qquad \forall \varphi \in \mathcal{D}(\Omega)$$
s.t. supp $\varphi \in K$

here equivalent K

then T is called distribution on Ω

$$\boxed{T \in \mathcal{D}'(\Omega)}$$

rem. equivalent $\forall K$ compact in Ω, $\exists \tilde{C}_K > 0$, $\exists m_K \in \mathbb{N}$
s.t.
$$|T(\varphi)| \leq \tilde{C}_K \cdot \sup_{|\alpha| \leq m_K} \left(\sup_{x \in \Omega} |\partial^\alpha \varphi(x)| \right)$$
$\forall \varphi \in \mathcal{D}(\Omega)$ with supp $\varphi \subseteq K$

rem. if $T \in \mathcal{D}'(\Omega)$
and $\exists m$ s.t. $m = m_K$ for all compact K in Ω,
the minimum of such m is called the order of T and T is said distribution of finite order

$$\mathcal{D}'_F(\Omega) = \{\text{distributions of finite order}\}.$$

Ex. 1. let Ω open set in $\mathbb{R}^d$

let $f \in L^1_{loc}(\Omega)$

$L^1_{loc}(\Omega) = \{ f$ measurable on Ω s.t. $\forall K$ compact $f \cdot \chi_K \in L^1(\Omega) \}$

ex. $\mathcal{C}(\Omega) \subseteq L^1_{loc}(\Omega)$

We associate to f a distribution T_f

$$\varphi \in \mathcal{D}(\Omega), \quad T_f(\varphi) = \int_\Omega f \varphi = \int_{\text{supp}\,\varphi} \underbrace{(f \cdot \chi_{\text{supp}\,\varphi})}_{\in L^1} \cdot \underbrace{\varphi}_{\in L^\infty}$$

¿it makes sense? (yes because the integration is made on the supp of φ which is a compact)

$T_f(\varphi)$ is well defined

it is linear (integral is linear)

fix K compact

$\text{supp}\,\varphi \subseteq K$

$$|T_f(\varphi)| = \left| \int_K f \cdot \varphi \right| \leq \int_\Omega |f \cdot \chi_K| \cdot |\varphi|$$

$$\leq \underbrace{\int_\Omega |f \chi_K|}_{C_K} \cdot \|\varphi\|_{L^\infty}$$

$$\boxed{|T(\varphi)| \leq C_K \sup_\Omega |\varphi| \quad \forall \varphi \in \mathcal{D}(\Omega) \text{ with supp } \varphi \subseteq K}$$

$T_f \in \mathcal{D}'(\Omega)$ with order 0

rem.

$$L^1_{loc}(\Omega) \xrightarrow{\phi} \mathcal{D}'(\Omega)$$

$$f \xmapsto{\phi} T_f$$

¿ is this functional ϕ injective? yes

$f_1, f_2 \in L^1_{loc}(\Omega)$ s.t. $T_{f_1} = T_{f_2}$

this means that $T_{f_1}(\varphi) = T_{f_2}(\varphi) \quad \forall \varphi \in \mathcal{D}$

$\int f_1 \varphi = \int f_2 \varphi$

$$T_{f_1 - f_2}(\varphi) = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$$

$$\int_\Omega (f_1 - f_2) \cdot \varphi = 0 \quad \forall \varphi \in \mathcal{C}^\infty_0(\Omega)$$

$$\Longrightarrow f_1 - f_2 = 0 \quad (\text{as } L^1_{loc} \text{ functions})$$

rem.

$L^1_{loc}(\Omega)$ can be thought as a sub set of $\mathcal{D}'(\Omega)$

Ex. $\mathcal{D}'(\Omega) \setminus L^1_{loc}(\Omega) \neq \emptyset$

consider $x_0 \in \Omega$.

take $\boxed{\delta_{x_0}(\varphi) = \varphi(x_0)}$

$\delta_{x_0} : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}$, it is linear

$\left(\delta_{x_0}(\varphi + \psi) = (\varphi + \psi)(x_0) = \varphi(x_0) + \psi(x_0) = \delta_{x_0}(\varphi) + \delta_{x_0}(\psi) \right)$

K compact and $\varphi \in \mathcal{D}(\Omega)$ and supp $\varphi \subseteq K$

$|\delta_{x_0}(\varphi)| = |\varphi(x_0)| \leq 1 \cdot \sup_{x \in \Omega} |\varphi(x)|$

δ_{x_0} is distribution of order 0.

"Dirac's delta at x_0"

$\delta_{x_0} \notin L^1_{loc}(\Omega)$ (as distributions)

suppose $\exists f \in L^1_{loc}$ s.t. $\underset{\substack{\| \\ \varphi(x_0)}}{\delta_{x_0}(\varphi)} = \int_\Omega f\varphi$, $\forall \varphi \in \mathcal{D}(\Omega)$

so taking $\psi \in C^\infty_0(\Omega \setminus \{x_0\})$ $(\subseteq C^\infty_0(\Omega))$

and then $\delta_{x_0}(\psi) = 0 = \psi(x_0)$

$\| $

$\int_\Omega f\psi \quad \Rightarrow \quad \int_\Omega f\psi = 0 \quad \forall \psi \in \mathcal{D}(\Omega \setminus \{x_0\})$

$\Downarrow$

$f = 0$ in $\Omega \setminus \{x_0\}$

$\Downarrow$

$f = 0$ in Ω (as L^1_{loc} function)

impossible because $\delta_{x_0} \neq 0$