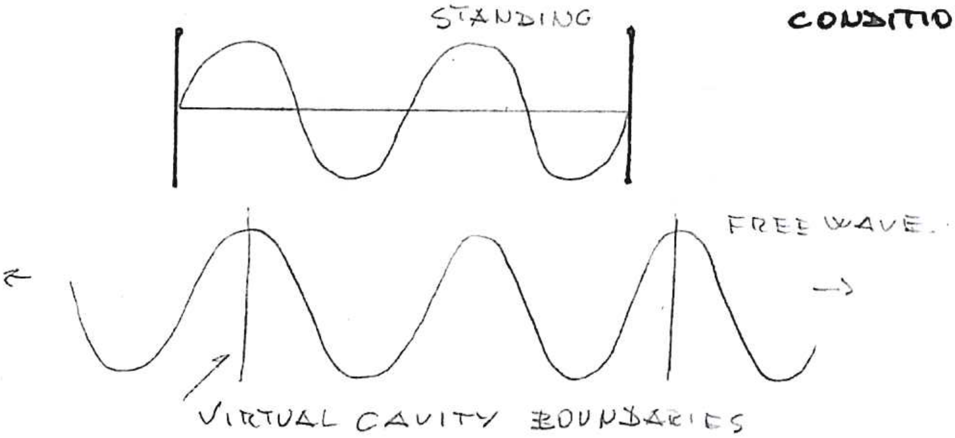
QUANTUM FORMULATION FOR THE E.M. FIELDS

LET US CONSIDER AN E.M. RADIATION IN A FREE SPACE WHERE,

$$J_T = 0 \quad \text{AND} \quad -\nabla^2 \bar{A} + \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} = 0 \qquad (166)$$

THE QUANTIZATION IS OBTAINED BY SUBSTITUTING THE VECTOR POTENTIAL $\bar{A}$ (CLASSICAL) WITH AN OPERATOR $\hat{A}$. TO DO SO WE CONSIDER A CUBIC SPACE VOLUME L^3. THIS VOLUME CAN BE CONSIDERED SIMILA TO A 3D OPTICAL RESONATOR BUT WITH NO MIRRORS. IN THESE CONDITIONS THE E.M. WAVES CAN BE REGARDED AS FREE WAVES INSTEAD OF STANDING WAVES, BUT SUBJECT TO THE BOUNDARIES CONDITIONS.



UNDER SUCH A CONDITIONS THE VECTOR POTENTIAL $\bar{A}$ CAN BE EXPANDED IN A FOURIER SERIE

$$\bar{A} = \sum_{K}{}' \left[\bar{A}_{\bar{k}}(t)\, e^{i\bar{k}\cdot\bar{r}} + \bar{A}_{\bar{k}}^{*}(t)\, e^{-i\bar{k}\cdot\bar{r}} \right] \qquad (167)$$

WHERE THE COMPONETS OF THE WAVEVECTOR $\bar{k}$ ASSUME THE VALUES

$$K_x = 2\pi\nu_x/L \; ; \quad K_y = 2\pi\nu_y/L \quad ; \; K_z = 2\pi\nu_z/L$$

BEING $\nu_x, \nu_y, \nu_z = 0, \pm 1, \pm 2 \ldots$

FROM WHICH WE CAN DEFINE THE MODES DENSITY.
THE COULOMB GAUGE IS ACCOMPLISHED IF

$$\bar{k} \cdot \bar{A}_k(t) = \bar{k} \cdot \bar{A}^*_k = 0 \qquad (168)$$

THIS IMPLIES THAT WE HAVE TWO INDEPENDENT DIRECTIONS OF $\bar{A}_k(t)$ FOR EACH $\bar{k}$.
THE FOURIER COMPONENTS OF $\bar{A}$ MUST BE INDEPENDENT AND MUST SATISFY THE FIELD EQUATIONS SUCH THAT

$$k^2 \bar{A}_k(t) + \frac{1}{c^2} \frac{\partial^2 A_{\bar{k}}(t)}{\partial t^2} = 0 \qquad (169)$$

AND OF COURSE $\bar{A}^*_{\bar{k}}(t)$ MUST SATISFY THE SAME CONDITIONS. THEREFORE, THE FOURIER COEFFICIENTS WILL SATISFY THE SIMPLE HARMONIC EQUATION:

$$\left(\partial^2 \bar{A}_k(t) / \partial t^2\right) + \omega^2_{\bar{k}} \bar{A}_k(t) = 0 \qquad (170)$$

WHERE $\omega_k = c|\bar{k}|$

THE QUANTIZATION OF THE E.M. FIELD IS OBTAINED IN ANALOGY OF THE QUATIZATION OF THE SIMPLE HARMONIC OSCILLATOR. TO DO SO WE RE-WRITE THE CLASSICAL OSCILLATOR EQUATION WITH THE POSITION AND MOMENTUM COORDINATES ASSOCIATED TO THE MODE OF THE CAVITY. FOR THIS REASON WE EVALUATE THE CLASSICAL ENERGY OF THE NORMAL MODE OF THE CAVITY IDENTIFIED BY THE VECTOR $\bar{k}$. THE SOLUTION OF THE ABOVE EQUATION CAN BE

$$\bar{A}_{\bar{k}}(t) = \bar{A}_k e^{-i\omega_{\bar{k}} t} \qquad (171)$$

AND THE COMPLETE VECTOR POTENTIAL $\bar{A}$ BECOMES

$$\bar{A} = \sum_{\bar{k}}{}' \left[\bar{A}_{\bar{k}} e^{(-i\omega_k t + i\bar{k}\cdot\bar{r})} + c.c.\right] \qquad (172)$$

THE MEAN ENERGY CALCULATED OVER A CYCLE FOR THE $\bar{k}$ MODE IS

$$\langle \mathcal{E}_{\bar{k}} \rangle = \frac{1}{2} \int_{CAVITY} \left(\varepsilon_0 \langle E_{\bar{k}}^2 \rangle + \frac{1}{\mu_0} \langle B_{\bar{k}}^2 \rangle \right) dV \quad (173)$$

WHERE $\bar{E}_{\bar{k}}$ AND $\bar{B}_{k}$ ARE THE ELECTRIC AND MAGNETIC FIELD RESPECTIVELY ASSOCIATED WITH THE MODE $\bar{k}$. FROM THE FIELD RELATIONS $\bar{B} = \bar{\nabla} \times \bar{A}$ AND $\bar{E} = -\nabla\phi - \partial\bar{A}/\partial t$ WE CAN OBTAIN THE FIELDS $\bar{E}$ AND $\bar{B}$ EXPRESSED IN FUNCTION OF $\bar{A}$

$$\bar{E}_{\bar{k}} = i\omega_k \left[\bar{A}_k e^{(-i\omega_k t + i\bar{k}\cdot\bar{r})} - c.c. \right]$$
$$\bar{B}_{\bar{k}} = i\bar{k} \times \left[\bar{A}_{\bar{k}} e^{(-i\omega_k t + i\bar{k}\cdot\bar{r})} - c.c. \right] \quad (174)$$

NOW IT IS POSSIBEE TO CALCULATE THE MEAN ENERGY AS

$$\langle \mathcal{E}_{\bar{k}} \rangle = 2\varepsilon_0 V \omega_{\bar{k}}^2 \bar{A}_{\bar{k}} \cdot \bar{A}_{\bar{k}}^*$$

WHERE $V = L^3$. ACTUALLY THE VARIABLES OF THE MODE CAN BE SUBSTITUTED BY THE GENERALIZED COORDINATES OF THE MODE Q_k AND P_k ACCORDINGLY WITH THE TRANSFORMATIONS

$$\bar{A}_{\bar{k}} = \left(4\varepsilon_0 V \omega_{\bar{k}}^2\right)^{-1/2} \left(\omega_{\bar{k}} Q_{\bar{k}} + i P_{\bar{k}}\right) \hat{\varepsilon}_k$$
$$\bar{A}_{\bar{k}}^* \quad \left(4\varepsilon_0 V \omega_{\bar{k}}^2\right)^{-1/2} \left(\omega_{\bar{k}} Q_{\bar{k}} - i P_{\bar{k}}\right) \hat{\varepsilon}_k \quad (175)$$

WHERE $Q_{\bar{k}}$ AND P_k ARE SCALARS AND THE DIRECTION OF THE VECTORS $\bar{A}_k$ AND $\bar{A}_k^*$ IS GIVEN BY THE POLARIZATION UNITARY VECTOR $\hat{\varepsilon}_{\bar{k}}$.

THE MODE AVERAGE ENERGY IS GIVEN BY

$$\langle \mathcal{E}_{\vec{k}} \rangle = \frac{1}{2}\left(P_{\vec{k}}^2 + \omega_{\vec{k}}^2 Q_{\vec{k}}^2\right) \quad (176)$$

AND THE COMPLETE CLASSICAL HAMILTONIAN FOR THE CAVITY IS GIVEN BY

$$H = 2 \sum_{k}' \langle \mathcal{E}_k \rangle \quad (177)$$

WHERE "2" STANDS FOR THE TWO POLARIZATION STATES. LET US RECALL NOW THE HAMILTONIAN OF THE 1-D Q.M.H. OSCILLATOR

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{q}^2\right) \quad (178)$$

WHERE $\hat{p}$ AND $\hat{q}$ ARE TWO OPERATORS THAT OBEY THE COMMUTATION

$$[\hat{q}, \hat{p}] = i\hbar .$$

WE DEFINE NOW TWO OPERATORS $\hat{a}$ AND $\hat{a}^+$ TO SUBSTITUTE $\hat{q}$ AND $\hat{p}$ AND DEFINED AS

$$\hat{a} = (2\hbar\omega)^{-1/2}(\omega\hat{q} + i\hat{p})$$
$$\hat{a}^+ = (2\hbar\omega)^{-1/2}(\omega\hat{q} - i\hat{p}) \quad (179)$$

THEREFORE

$$\hat{q} = (\hbar/2\omega)^{+1/2}(\hat{a} + \hat{a}^+)$$
$$\hat{p} = -i(\hbar\omega/2)^{+1/2}(\hat{a} - \hat{a}^+) \quad (180)$$

$\hat{a}$ AND $\hat{a}^+$ REPPRESENT THE OPERATOR

ANNIHILATION AND CREATION, RESPECTIVELY FOR THE Q.H.O.

$\hat{a}$ AND $\hat{a}^+$ ARE NOT OBSERVABLE

$$\hat{a}^+\hat{a} = (2\hbar\omega)^{-1}(\hat{p}^2 + \omega^2\hat{q}^2 + i\omega\hat{q}\hat{p} - i\omega\hat{p}\hat{q})$$
$$= (\hbar\omega)^{-1}\left(\hat{H} - \tfrac{1}{2}\hbar\omega\right) \tag{181}$$

SIMILARLY IT IS POSSIBLE TO OBTAIN

$$\hat{a}\,\hat{a}^+ = (\hbar\omega)^{-1}\left(\hat{H} + \tfrac{1}{2}\hbar\omega\right) \tag{182}$$

$$\Downarrow$$

$$[\hat{a}, \hat{a}^+] = \hat{a}\,\hat{a}^+ - \hat{a}^+\hat{a} = 1 \tag{183}$$

WE CAN WRITE THE HAMILTONIAN AS

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \tfrac{1}{2}\right) \tag{184}$$

THE TERM $\hat{a}^+\hat{a}$ IS ALSO KNOWN AS NUMBER OPERATOR

$$\hat{n} = \hat{a}^+\hat{a}$$

TO PROVE THIS LET US CONSIDER $|n\rangle$ AS AN EIGENSTATE OF THE H.O. WITH AN EIGENVALUE E_n. THE EIGENVALUE EQUATION IS

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^+\hat{a} + \tfrac{1}{2}\right)|n\rangle = E_n|n\rangle \tag{185}$$

BY MULTIPLING BOTH SIDES BY $\hat{a}^+$ (FROM LEFT)

$$\hbar\omega\left(\hat{a}^+\hat{a}^+\hat{a} + \tfrac{1}{2}\hat{a}^+\right)|n\rangle = E_n\hat{a}^+|n\rangle.$$

BY USING THE COMMUTATIVE RELATION FOR THE LEFT-SIDE TERM, WE OBTAIN

$$\hbar\omega\left(\hat{a}^+\hat{a}\,\hat{a}^+ - \hat{a}^+ + \tfrac{1}{2}\hat{a}^+\right)|n\rangle = E_n\,\hat{a}^+|n\rangle \tag{186}$$

THAT CAN BE RE-WRITTEN AS:

$$\hbar\omega\left(\hat{a}^+\hat{a}+\frac{1}{2}\right)\hat{a}^+|n\rangle=\hat{H}\,\hat{a}^+|n\rangle=(E_n+\hbar\omega)\hat{a}^+|n\rangle \quad (187)$$

ALSO THIS IS AN EIGENVALUE EQ. THAT SHOWS HOW APPLYING THE OPERATOR $\hat{a}^+$ TO THE PREVIOUS EIGENVALUE EQ. WE OBTAIN THAT $\hat{a}^+|n\rangle$ IS STILL AN EIGENSTATE WHICH EIGENVALUE IS $E_n+\hbar\omega$. LET US DEFINE NOW THIS NEW EIGENSTATE AND EIGENVALUE AS

$$|n+1\rangle=\hat{a}^+|n\rangle \; ; \; E_{n+1}=E_n+\hbar\omega \quad (188)$$

THEREFORE THE PREVIOUS EQUATION CAN BE WRITTEN AS

$$\hat{H}|n+1\rangle=E_{n+1}|n+1\rangle \quad (189)$$

THIS PROVES THAT APPLAYING THE OPERATOR $\hat{a}^+$ TO A MODE WHICH EIGENVALUE IS E_n WE OBTAIN A MODE WHICH EIGENVALUE IS $E_n+\hbar\omega=E_{n+1}$. THE OPPOSITE IS OBTAINED BY APPLYING THE OPERATOR $\hat{a}$, i.e. $E_n-\hbar\omega=E_{n-1}$

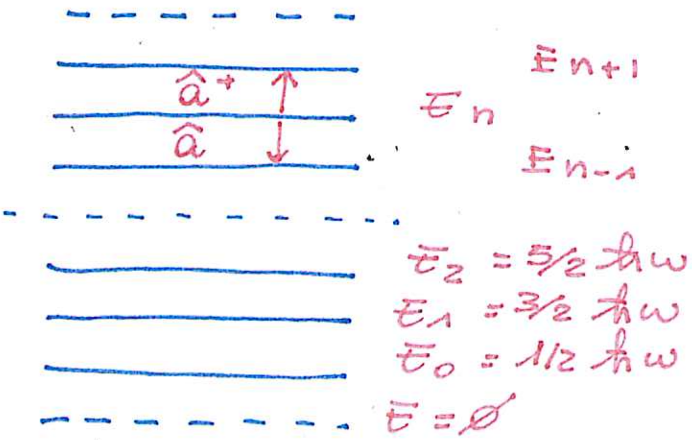
$$\hat{H}\,\hat{a}|n\rangle=(E_n-\hbar\omega)\hat{a}|n\rangle$$

AND BY USING CONSISTENT DEFINITION AS BEFORE WE OBTAIN

$$\hat{H}|n-1\rangle=E_{n-1}|n-1\rangle \quad (190)$$

IT IS POSSIBLE TO VISUALIZE THIS RESULT USING A "LADDER" REPPRESENTATION. HOWEVER IT MUST BE POINTED OUT THAT WHILE THE OPERATOR $\hat{a}^+$ CAN BE APPLIED INDEFINITELY, $\hat{a}$ MUST HAVE AN INFERIOR LIMIT. IF $|0\rangle$ IS THE FUNDAMENTAL STATE WE OBTAIN

$\hat{H}\hat{a}|\emptyset\rangle = (E_0 - \hbar\omega)\hat{a}|\emptyset\rangle$. SINCE THERE ARE NOT EIGENSTATES WITH A MINOR ENERGY THE ONLY SOLUTION WITH PHYSICAL MEANING IS $\hat{a}|\emptyset\rangle$.



THE E.M. FIELD IS NOW QUANTIZED BY THE ASSOCIATION OF A Q.M.H. TO THE MODE $\bar{k}$ OF THE E.M. FIELD. THE CLASSICAL VECTORS $\bar{A}_{\bar{k}}$ AND $\bar{A}^{+}_{\bar{k}}$, AS DEFINED BEFORE FOR THE $\bar{k}$ MODE AND REPPRESENTED BY $P_{\bar{k}}$ AND $Q_{\bar{k}}$ CAN BE CONVERTED IN Q.M. OPERATORS AND EXPRESSED BY THE OPERATORS $\hat{p}_{\bar{k}}$ AND $\hat{q}_{k}$. THIS SUBSTITUTION LEADS TO THE FOLLOWING EQUATIONS

$$\bar{A}_{\bar{k}} = \left(4\varepsilon_0 V \omega^2_{\bar{k}}\right)^{-1/2}\left(\omega_{\bar{k}} \bar{Q}_{\bar{k}} + i \bar{P}_{\bar{k}}\right)\hat{\varepsilon}_{\bar{k}}$$

$$\Downarrow$$

$$= \left(4\varepsilon_0 V \omega^2_{\bar{k}}\right)^{-1/2}\left(\omega_{\bar{k}} \hat{q}_{\bar{k}} + i \hat{p}_{\bar{k}}\right)\hat{\varepsilon}_{\bar{k}}$$

$$\Downarrow$$

$$= \left(\hbar / 2\varepsilon_0 V \omega_{\bar{k}}\right)^{1/2} \hat{a}_{\bar{k}} \hat{\varepsilon}_{\bar{k}}$$

$$\bar{A}^{*}_{\bar{k}} = \left(\hbar / 2\varepsilon_0 V \omega_{\bar{k}}\right)^{1/2} \hat{a}^{+}_{\bar{k}} \hat{\varepsilon}_{\bar{k}}$$

THEREFORE, THE VECTOR POTENTIAL OPERATOR CAN BE WRITTEN AS:

$$\hat{A} = \sum_{\bar{k}}{}' \left(\hbar/2\varepsilon_0 V\omega_{\bar{k}}\right)^{1/2} \hat{\varepsilon}_{\bar{k}} \left\{\hat{a}_{\bar{k}} e^{-i\omega_{\bar{k}}t+i\bar{k}\cdot r} + \hat{a}^{+}_{\bar{k}} e^{+i\omega_{\bar{k}}t-i\bar{k}\cdot\bar{r}}\right\}$$

FROM WHICH IS POSSIBLE TO OBTAIN THE FIELDS, $\hat{E}$ AND $\hat{B}$, OPERATORS:

(192)
$$\hat{E}_{\bar{k}} = i\left(\hbar\omega_{\bar{k}}/2\varepsilon_0 V\right)^{1/2} \hat{\varepsilon}_{k} \left\{\hat{a}_{\bar{k}} e^{-i\omega_{k}t+i\bar{k}\cdot\bar{r}} + \hat{a}^{+}_{k} e^{+i\omega_{\bar{k}}t-i\bar{k}\cdot\bar{r}}\right\}$$

$$\hat{B}_{\bar{k}} = i\left(\hbar/2\varepsilon_0 V\omega_{\bar{k}}\right)^{1/2} \times\hat{\varepsilon}_{k} \left\{\hat{a}_{\bar{k}} e^{-i\omega_{\bar{k}}t+i\bar{k}\cdot\bar{r}} + \hat{a}^{+}_{\bar{k}} e^{+i\omega_{\bar{k}}t-i\bar{k}\cdot\bar{r}}\right\}$$

$$\Downarrow$$

$$\hat{E}_{T} = \sum_{\bar{k}}{}' \hat{E}_{k} \quad ; \quad \hat{B} = \sum_{\bar{k}}{}' \hat{B}_{\bar{k}}$$

MAKING USE OF THE FIELDS OPERATORS TO CALCULATE THE ENERGY THE $\bar{k}$-MODE IS EXCITED WE NEED TO CALCULATE THE NUMBER OF PHOTONS IN THE MODE

(193)
$$\langle \varepsilon_{\bar{k}} \rangle = \frac{1}{2}\int_{CAV.} \langle \langle n_{\bar{k}} | \varepsilon_0 \hat{E}_{\bar{k}}\cdot\hat{E}_{k} + \mu_0^{-1} \hat{B}_{k}\cdot\hat{B}_{k} | n_{\bar{k}} \rangle\rangle dV$$

WHERE THE AVERAGE IS OVER ONE CYCLE. THE HAMILTONIAN FOR THE ENTIRE E.M. FIELD IS GIVEN BY THE SUMMATION OF THE TERMS

(194)
$$\hat{H} = \hbar\omega\left(\hat{a}\hat{a}^{+} + 1/2\right) \Rightarrow \hat{H}_{T} = \sum_{\bar{k}}{}' \hbar\omega_{k}\left(\hat{a}\hat{a}^{+} + 1/2\right)$$

THE TOTAL ENERGY

$$\langle \varepsilon \rangle = \sum_{\bar{k}}{}' \langle \varepsilon_{\bar{k}} \rangle = \sum_{\bar{k}} \left(n_{\bar{k}} + 1/2\right)\hbar\omega_{k}$$