$X \subseteq \mathbb{A}^n_K$ $\quad K[X] = K[x_1, \dots, x_n]/I(X)$ coordinate ring of X

If K is algebraically closed : bijection

$\{$prime ideals of $K[X]\} \longleftrightarrow \{$irred. subvarieties of $X\}$

$\dim X = \dim K[X]$ $\qquad K[X] = K[t_1, \dots, t_n]$

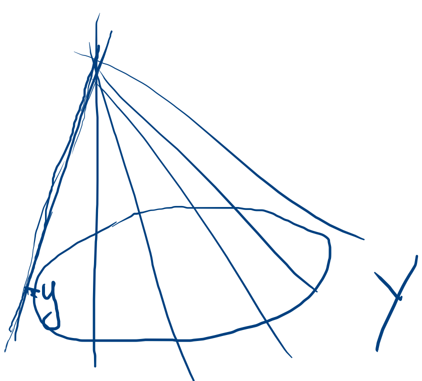
$Y \subseteq \mathbb{P}^n_K$ $\quad I_h(Y)$ $\qquad \dfrac{K[x_0, \dots, x_n]}{I_h(Y)} = S(Y)$

homogeneous coordinate ring Y

The elements of $S(Y)$ are $[F]$, not functions on Y.

$I_h(Y) = I(C(Y))$ $\qquad C(Y) \subseteq \mathbb{A}^{n+1}_K$

$S(Y) = K[C(Y)]$ polynomial functions on $C(X)$



$p: C(Y) \setminus \{0\} \longrightarrow Y$

$p^{-1}(y)$ $\qquad y$

line through 0

$$S(Y) = \frac{K[x_0, \dots, x_n]}{I_a(Y)}$$ it is canonically graded

$$S(Y) = \bigoplus_{d \geq 0} S(Y)_d \qquad S(Y)_d = \frac{K[x_0 \dots x_n]_d}{I_a(Y)_d}$$

$$[F] = [F_0 + F_1 + \dots + F_d] = [F_0] + [F_1] + \dots + [F_d]$$
$$\parallel$$
$$[G] = [G_0 + G_1 + \dots + G_e] = [G_0] + [G_1] + \dots + [G_e]$$

$F - G \in I_a(Y)$ homogenous

$$\parallel$$

$$(F_0 - G_0) + (F_1 - G_1) + \dots + (F_d - G_d) + \dots G_e \qquad \text{if } e \geq d$$

$$\Rightarrow F_0 - G_0, F_1 - G_1, \dots, F_d - G_d, \dots, G_e \in I_a(Y)$$

$$\Rightarrow [F_0] = [G_0], \ [F_1] = [G_1], \dots$$

unicity : the sum is direct.

$\mathbb{P}^n_K$ $\dim \mathbb{P}^n_K = \dim \mathbb{A}^n_K$ K infinite

$$S(\mathbb{P}^n_K) = K[x_0 \dots x_n] / (0) = K[x_0 \dots x_n]$$

If K alg. closed $\dim K[x_0, \dots, x_n] = \dim \mathbb{A}^{n+1}_K$

$$\Rightarrow \dim \mathbb{P}^n \neq \dim K[x_0, \dots, x_n] = \dim S(\mathbb{P}^n_K)$$

<u>Theorem</u> <u>K</u> any field, A finitely generated K-algebra integral domain. Then

1) $\mathrm{tr.d.}\ Q(A)/K = \dim A$

In particular: since $\mathrm{tr.d.}\ Q(A)/K = \mathrm{tr.d.}\ Q(K[t_1, \dots, t_n])/K =$

$$= \mathrm{tr.d.}\ K(t_1, \dots, t_n)/K \leq n$$

So $\dim K[t_1, \dots, t_n] \leq n$.

2) $P \subseteq A$ prime ideal. Then

$$\dim A = \dim A/P + \boxed{\mathrm{ht}\, P}$$

$\mathrm{ht}\, P$ <u>heigth of P</u> = sup of the lengths of chains of prime ideals contained in P

$$\underbrace{\dim A}_{\text{chains of prime ideals in } A} = \underbrace{\dim A/P}_{\text{chains of prime ideals of } A \text{ containing } P} + \underbrace{\operatorname{ht} P}_{\text{ch. of prime ideals contained in } P}$$

$$m \qquad \underbrace{P_0 \subsetneq \cdots \subsetneq \overset{}{\underset{\underset{P_r}{\parallel}}{P}} \subsetneq \cdots \subsetneq P_m}$$

$\exists$ a chain of max length passing through P

Consequences of the theorem: K algebraically closed

1) $\dim \mathbb{A}^n_K = n$

$$\dim \mathbb{A}^n_K = \dim K[x_1, \dots, x_n] = \operatorname{tr.d.} K(x_1, \dots, x_n)/K = n$$

$$\dim \mathbb{P}^n_K = n$$

2) $X \subseteq \mathbb{A}^n_K$ affine variety, <u>X irreducible</u>

$$K[X] = \frac{K[x_1 \ldots x_n]}{I(X)} \text{ is an integral domain}$$

prime

$$\| $$

$$K[t_1, \ldots, t_n]$$

$$\underline{\dim X} = \dim K[X] = \text{tr. d.} \frac{Q(K[X])}{K} = \text{tr. d.} \frac{K(t_1, \ldots, t_n)}{K}$$

Def. $Q(K[X])$ field of <u>rational functions on X</u>

$$\| $$

$$K(X)$$

$$K(X) = \left\{ \frac{f}{g} \;\middle|\; f, g \in K[X], g \neq 0 \right\}$$

$\frac{f}{g}$ is a function on X, defined by a quotient of 2 polynomials : $f = [F]$, $g = [G]$

$\frac{f}{g}$ is represented by $\frac{F}{G}$

<u>Attention</u> : $g \neq 0$ $\quad G \notin I(X)$:

<u>$\exists P \in X$ s.t. $G(P) \neq 0$</u>

It is not true that $G(P) \neq 0$ $\forall P \in X$

$V(G) \cap X$ can be $\neq \emptyset$

$\frac{f}{g}$ is a function only at points $P \in X$ where $g(P) \neq 0$

3) $X \subseteq \mathbb{A}^n_K$ irreducible, $I(X)$ prime

$$\dim X = \dim K[X] = n - \boxed{\mathrm{ht}\, I(X)}$$

$$n = \dim K[x_1 \ldots x_n] = \dim \frac{K[x_1 \ldots x_n]}{I(X)} + \mathrm{ht}\, I(X)$$

$\dim X \leq n$

$L \subseteq \mathbb{A}^n$ L is the set of solutions of a linear system $\dim L = n - \mathrm{rk}\, M$

M matrix of the coeff. of the lin. system

Consequence for hypersurfaces, K algebraically closed field

$X \subseteq \mathbb{A}^n_K$ affine algebraic set

X is a hypersurface $\iff$ X has pure dimension $n-1$:

$X = V(F)$: $\{F = 0$ all irreducible components of X

F square-free polyn. have dimension $n-1$

Pf "$\Rightarrow$" X $\quad I(X) = (F)$ $\quad F = F_1 F_2 \cdots F_r$, $F_1, \dots, F_r$ irreducible

$$X = V(F_1 \cdots F_r) = \underbrace{V(F_1)}\cup \cdots \cup \underbrace{V(F_r)}.$$

irreducible components of X

$V(F_i) \not\subseteq V(F_j)$: otherwise $\underset{(F_i)}{\underset{\|}{I(V(F_i))}} \supseteq \underset{(F_j)}{\underset{\|}{I(V(F_j))}} \Rightarrow F_j = H F_i$.

$\Rightarrow H$ invertible $\Rightarrow (F_i) = (F_j)$: no

$$\dim X = \sup_{i=1,\dots,r} \dim V(F_i)$$

claim: $\dim V(F_i) = n-1 \iff \operatorname{ht}(F_i) = 1$

The claim is that: if F_i is an irred. polyn.

$\underline{\mathrm{lt}(F_i)=1}$ $\quad \underset{\text{prime}}{P} \subseteq (F_i)$

$\underline{P=(0)}$ prime

$P \neq (0)$: $\exists\, G \in P$, $G \neq 0$

at least one irred. factor of G belongs to P, prime $\Longrightarrow$ P contains an irred. pol. $G \in P$

$(G) \subseteq P \subseteq (F_i)$ $\qquad \underset{\text{irred.}}{\overline{G}} = AF_i \Rightarrow A$ unit

$(G) = (F_i) = P$

$\Rightarrow (0) \subsetneq (F_i)$ $\qquad \mathrm{lt}(F_i)=1$

"$\Longleftarrow$" X has pure dim $n-1$

$X = X_1 \cup \dots \cup X_r$ $\qquad \dim X_i = n-1 \ \forall i$

irred. components of X

$\underset{n-1}{X_i} \subsetneq \underset{n}{\mathbb{A}^n} \Rightarrow \underset{\text{prime}}{\underline{I}(X_i)} \neq (0)$: $\exists\, F$ irred. $F \in \underline{I}(X_i)$

$V(F) \supseteq X_i$ $\Longrightarrow$ $X_i = V(F)$

$\underline{\text{irred}}$ hypersurf. $n-1$ $\qquad$ closed $n-1$

dim is finite

$X = V(F_1) \cup \dots V(F_r) =$

$= V(F_1 \cdots F_r)$

hypersurface $\square$

"Pure dimension $n-1$" = "$\underline{\text{one equation}}$"

$\underline{\text{codimension } 1}$

For codimension $r>1$: the codimension is not the number of equations $\underline{\text{in general}}$

In the proj. space $\underset{3}{\mathbb{P}^3_x} \supseteq \underset{1}{\overline{X}}$ skew cubic

$\boxed{I_n(\overline{X})}$ it is generated by 3 polynomials

$\dim \overline{X}$ $\overline{X}$ is the proj. closure of the aff. skew cubic X

X is homeom. to $\mathbb{A}^1$: $\dim X = 1$

$\dim \overline{X} = 1$

$\operatorname{codim}_{\mathbb{P}^3} \overline{X} = 2$: we need 3 equations for $\overline{X}$

$\underline{I}(X) = (y - x^2, z - x^3)$: in this case

2 equations describe X

$\operatorname{codim}_{\mathbb{A}^3} X$

$$\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^3 \qquad \varphi(\mathbb{A}^1) = Y =$$
$$t \longrightarrow (t^3, t^4, t^5) \qquad = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in \mathbb{A}^1\}$$

$\underline{I}(Y) = \langle \underline{x^3 - yz,\ y^2 - xz,\ z^2 - x^2 y} \rangle$ K infinite

some work

$$\varphi : t \longrightarrow (t^m, t^n, t^p)$$

<u>Prop.</u> Dimension of a product of 2 irred. affine varieties.

$X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ both irreducible $\Rightarrow$ $X \times Y \subseteq \mathbb{A}^{n+m}$ irreducible

K alg. closed

$\dim(X \times Y) = \dim X + \dim Y$

<u>Pf</u>: $r = \dim X$, $s = \dim Y = \dim K[Y]$
$\quad\quad \| $
$\quad \dim K[X]$

$K[X] = K[t_1, \ldots, t_n]$, $t_1, \ldots, t_n$ coord. fns on X

$K[Y] = K[u_1, \ldots, u_m]$, $u_1, \ldots, u_m$ coord. fns on Y

$r = \text{tr.d.} \dfrac{K(t_1, \ldots, t_n)}{K}$: we can renumber the coordinates so that $t_1, \ldots, t_r$ are alg. indep. over K

$s = \text{tr.d.} \dfrac{K(u_1, \ldots, u_m)}{K}$, $u_1, \ldots, u_s$ are alg. indep. - - K

$$K[X \times Y] = \frac{K[x_1 \ldots x_n, y_1 \ldots y_m]}{I(X \times Y)} = K[t_1 \ldots t_n, u_1 \ldots u_m]$$

<u>claim</u> $t_1 \ldots t_r, u_1, \ldots, u_s$ is a transcendence basis for $K(X \times Y)$

1) $t_1 \dots t_r, u_1, \dots, u_s$ are alg indep over K

Ans. F poly in $r+s$ variables vanishing on $t_1 \dots t_r, u_1 \dots u_s$

$F(t_1 \dots t_r, u_1 \dots u_s) = 0$: this is a function on $X \times Y$

$\forall P \in X$, $P = (a_1 \dots a_r)$: $F(a_1 \dots a_r, u_1 \dots u_s) = 0$ as a function on Y

$u_1, \dots, u_s$ are alg indep $\Rightarrow F(a_1 \dots a_r, Y_1, \dots, Y_s) = 0$

as polyn in $Y_1, \dots, Y_s$ $\Rightarrow$ all the coeff of $\uparrow$ are 0

The coeff are polynomials in $x_1, \dots, x_r$, vanishing

$\forall P \in X$ $\Rightarrow$ are functions in $t_1, \dots, t_r$ which are

0 in $K[X]$, alg indep.

$\Rightarrow$ the coeff of $F(x_1 \dots x_r, Y_1 \dots Y_s)$ as pol

in $Y_1 \dots Y_s$ are 0 $\Longrightarrow$ $F \equiv 0$ $\Longrightarrow$

$t_1 \dots t_r, u_1, \dots, u_s$ are alg indep.

2) $t_1 \dots t_r, u_1 \dots u_s$ is a max set of alg indep.

elements in $K[X \times Y]$: it follows from the fact

that $t_1, \dots, t_r$ is a max set of alg indep elem in $K[X]$

$u_1 \dots u_s$ $\qquad$ " $\qquad$ $K[Y]$

$K[X \times Y] = K[\underbrace{t_1 \dots}_{r} t_n, u_1 \dots u_m]$

$t_{r+1}, \dots, t_n$ are algebraic over $K(t_1 \dots t_r)$ } over $K(t_1 \dots t_r, u_1 \dots u_s)$

$u_{s+1}, \dots, u_m$ " " " $K(u_1 \dots u_s)$

$\Longrightarrow$ $t_1 \dots t_r\, u_1 \dots u_s$ is a transc basis.