Distributions.

def. $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) $\qquad \mathcal{D}(\Omega) = C_0^\infty(\Omega)$

test functions on Ω

T linear

T is a distribution if

$\forall K$ compact set in Ω, $\exists\, C_K > 0$, $m_K \in \mathbb{N}$

s.t. $$|T(\varphi)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_{x \in \Omega} |\partial^\alpha \varphi(x)|$$

$\forall \varphi \in \mathcal{D}(\Omega)$ s.t. supp $\varphi \subseteq K$.

It is a sort of "continuity condition on T"

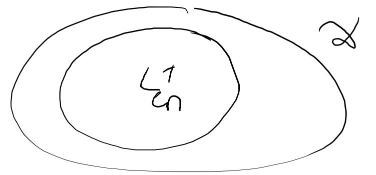
Ex. let $f \in L^1_{loc}(\Omega)$

define T_f s.t. $T_f(\varphi) = \int_\Omega f \cdot \varphi$

then $T_f \in \mathcal{D}'(\Omega)$

moreover $T_{f_1} = T_{f_2}$ in $\mathcal{D}'(\Omega)$ $\iff$ $f_1 = f_2$ in $L^1_{loc}(\Omega)$

rem. we can think at $L^1_{loc}(\Omega)$ as a subspace of $\mathcal{D}'(\Omega)$



Ex. $\mathcal{D}'(\Omega) \setminus L^1_{loc}(\Omega) \neq \emptyset$

example take $x_0 \in \Omega$,

define $\delta_{x_0}(\varphi) = \varphi(x_0)$

then δ_{x_0} (the Dirac's delta) is a distribution (of order 0)

rem. if m_K in def of distribution does not depend on K, the minimum value of m_K is called the order of the distribution.

$f \in L^1_{loc}(\Omega) \Rightarrow T_f$ is of order 0

and δ_{x_0} is of order 0.

Ex. let $\Omega =]-1, 1[$

consider $\text{dip} : \mathcal{D}(]-1,1[) \to \mathbb{R}$

$\varphi \longmapsto \varphi'(0)$

operator "dipole"

obviously dip is linear

and $|\text{dip}(\varphi)| = |\varphi'(0)| \leq 1 \cdot \sum_{j \leq 1} \sup_{x \in]-1,1[} |\varphi^{(j)}(x)|$ order = 1

$(\forall \varphi \in \mathcal{C}_0^\infty(]-1,1[))$

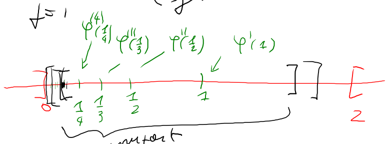
both C_K and m_K do not depend on K

Ex $\exists$ distributions which are not distributions of finite order (we call them of infinite order)

Consider $\Omega =]0, 2[$

consider $T(\varphi) = \sum_{j=1}^{+\infty} \varphi^{(j)}(\frac{1}{j})$ ⊛

$\varphi \in \mathcal{C}_0^\infty(]0,2[)$



is ⊛ a series? no it is a finite sum.

because if $\varphi \in \mathcal{C}_0^\infty(]0,2[)$

then $\exists\, \bar{k} \in \mathbb{N}$ s.t. $\underbrace{\text{supp}\,\varphi}_{\text{is compact}} \subseteq [\frac{1}{\bar{k}}, 2 - \frac{1}{\bar{k}}]$ in $]0,2[$

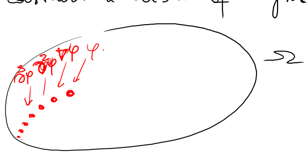
so that $T(\varphi) = \sum_{j=1}^{\bar{k}} \varphi^{(j)}(\frac{1}{j})$

and $|T(\varphi)| \leq C_K \sum_{j \leq \bar{k}} \sup_{x \in]0,2[} |\varphi^{(j)}(x)|$

for all φ with support in $[\frac{1}{\bar{k}}, 2-\frac{1}{\bar{k}}]$

but the value $\bar{k}$ depends on the compact

idea to construct a dist. of infinite order on Ω



sequence of points that goes to $\partial\Omega$ and evaluate derivatives of φ of increasing order on these points

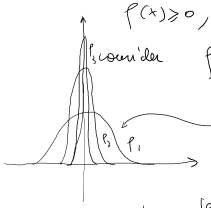
def. Let $(T_n)_n$ a sequence of distributions in $\mathcal{D}'(\Omega)$.
Suppose that $T \in \mathcal{D}'(\Omega)$.

we will say that $\lim_n T_n = T$
when $\forall \varphi \in \mathcal{D}(\Omega)$, $\lim_n T_n(\varphi) = T(\varphi)$ | convergence in the sense of $\mathcal{D}'$,

(it is the "weak" convergence)

Ex. consider $\rho \in C_0^\infty(\mathbb{R}^d)$, supp $\rho \subseteq B(0,1)$,
$\rho(x) \geq 0$, $\int \rho(x)\,dx = 1$

consider $\rho_n(x) = n^d \rho(nx)$ $\qquad (\rho_n)_n$ mollifier



prove that $\lim_n T_{\rho_n} = \delta_0$

I have to prove that, $\forall \varphi \in C_0^\infty(\mathbb{R}^d)$

$$\lim_n T_{\rho_n}(\varphi) = \delta_0(\varphi) = \varphi(0)$$

$$T_{\rho_n}(\varphi) = \int_{\mathbb{R}^d} \rho_n(y)\varphi(y)\,dy = \int_{|y| \leq \frac{1}{n}} n^d \rho(ny)\varphi(y)\,dy$$

$ny = x$
$n^d dy = dx$

$$= \int_{|x| \leq 1} \underbrace{\rho(x)\,\varphi\left(\frac{x}{n}\right)}_{f_n(x)} dx$$

I want to use the dominated convergence theorem

$f_n(x) \xrightarrow{n \to +\infty} \rho(x)\varphi(0)$ $\qquad$ pointwise

$$|f_n(x)| = |\rho(x)\varphi(\tfrac{x}{n})| \leq \rho(x) \cdot \underbrace{\max|\varphi(\tfrac{x}{n})|}_{\|\varphi\|_{L^\infty}}$$

$(\rho(x) \geq 0)$

so $|f_n| \leq \rho \cdot \underset{\|\varphi\|_{L^\infty}}{\overset{\shortparallel}{\text{constant}}}$

$$\lim_n \int \rho_n(x)\varphi(x)\,dx = \int \rho(x)\cdot\varphi(0)\,dx = \varphi(0)\cdot \underbrace{\int\rho(x)\,dx}_{=1} = \varphi(0)$$

OK

remark. δ_0 is the limit (as distribution) of a sequence of functions which are 0 outside $B(0,\frac{1}{n})$ and which have integral $= 1$

so δ_0 "is a function which is 0 outside 0 and the integral is 1" remark: it is possible to put a topology on $\mathcal{D}(\Omega)$ in such a way that $\mathcal{D}'(\Omega)$ is the dual space

complicated!

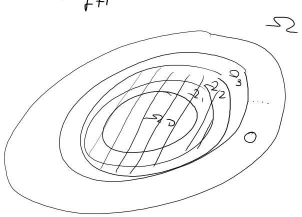
idea. consider $\mathcal{C}^m(\Omega)$

Ω open set topology of unif. with the convergence on compacts of functions and derivatives up to order m

take $\Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \dots$ open sets

s.t. $\overline{\Omega_j}$ is compact in Ω_{j+1}

and $\bigcup_{j=0}^{+\infty} \Omega_j = \Omega$



take

$$p_j(f) = \sum_{|\alpha| \le m} \sup_{x \in \Omega_j} |\partial^\alpha f(x)|$$

p_j is seminorm (it is not a norm)

Consider the topology on $\mathcal{C}^m(\Omega)$ obtained by the metric

$$d(f,g) = \sum_{j=1}^{+\infty} \frac{1}{2^j} \frac{p_j(f-g)}{1+p_j(f-g)}$$

distance on $\mathcal{C}^m(\Omega)$ — complete metric space (Fréchet space)

topology comes from a sequence of seminorms

Similar construction for the topology of $\mathcal{C}^\infty(\Omega)$

$$\tilde{p}_j(f) = \sum_{|\alpha| \le j} \sup_{x \in \Omega_j} |\partial^\alpha f(x)|$$

take $\mathcal{C}_0^\infty(\overline{\Omega}_j)$ closed space in $\mathcal{C}^\infty(\Omega)$

consider $\mathcal{C}_0^\infty(\overline{\Omega}_j) \hookrightarrow \mathcal{C}_0^\infty(\Omega)$

take here the maximal topology for which all this inclusions are continuous

see BOURBAKI

see also TREVES

Theorem (characterisation of distributions)

Let Ω open set

Suppose $T : \mathcal{D}(\Omega) \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) linear

then T is distribution if and only if

(*) for every $(\varphi_n)_n$ sequence in $\mathcal{D}(\Omega)$ such that

1) $\exists K$ compact s.t. $\forall n$, $\operatorname{supp} \varphi_n \subseteq K$

2) $\forall \alpha \in \mathbb{N}^n$, $\partial^\alpha \varphi_n \longrightarrow 0$ uniformly

we have $\lim_n T(\varphi_n) = 0$

proof

if T is distribution then (*) is valid. In fact

T distribution take $(\varphi_n)_n$ with the properties 1) and 2)

in particular $\operatorname{supp} \varphi_n \subseteq K$ ($\exists K$ s.t. for all n ....)

Plug K in definition of T

$\exists C_K, m_K$ s.t.

$$|T(\varphi)| \leq C_K \cdot \sum_{|\alpha| \leq m_K} \sup_{x \in \Omega} |\partial^\alpha \varphi| \qquad \forall \varphi \text{ with supp. in } K$$

in particular

$$|T(\varphi_n)| \leq C_n \sum_{|\alpha| \leq m_K} \underbrace{\sup_{x \in \Omega} |\partial^\alpha \varphi_n|}_{\text{this quantity is going to } 0}$$

and then $\lim_n T(\varphi_n) = 0$

Conversely suppose (*) is valid and, by contradiction,

$T \notin \mathcal{D}'(\Omega)$

$\exists K$ compact s.t. $\forall C > 0$, $m \in \mathbb{N}$, $\exists \varphi_{C,m} \in \mathcal{D}(\Omega)$

with $\operatorname{supp} \varphi_{C,m} \subseteq K$ and

$$|T(\varphi_{C,m})| > C \cdot \sum_{|\alpha| < m} \sup_{x \in \Omega} |\partial^\alpha \varphi_{C,m}|$$

In particular take $C = m = j$ then $\forall j$ $\exists \varphi_j$ s.t.

$\operatorname{supp}(\varphi_j) \subseteq K$ and $|T(\varphi_j)| > j \cdot \sum_{|\alpha| \leq j} \sup_{x \in \Omega} |\partial^\alpha \varphi_j|$

Consider
$$\psi_j(x) = \frac{\varphi_j(x)}{j \cdot \sum_{|\alpha| \leq j} \sup_{y \in \Omega} |\partial^\alpha \varphi_j(y)|}$$

consider $(\psi_j)_j \longrightarrow$ for all j, $\operatorname{supp} \psi_j \subseteq K$

take $\beta \in \mathbb{N}$ consider $\partial^\beta \psi_j(x) = \dfrac{\partial^\beta \varphi_j(x)}{j \cdot \sum_{|\alpha| \leq j} \sup_\Omega |\partial^\alpha \varphi_j|}$

for j $j > |\beta|$

then $|\partial^\beta \psi_j(x)| \leq \dfrac{1}{j}$ $\forall x \in \Omega$

$\psi_j \to 0$ uniformly with all its derivatives.

the sequence $(\psi_j)_j$ satisfies 1) and 2)

but $\lim_j T(\psi_j) \neq 0$

since $|T(\psi_j)| = \dfrac{|T(\varphi_j)|}{j \sum_{|\alpha| \leq j} \sup |\partial^\alpha \varphi_j|} > 1$

(*) is not valid

impossible

Local character of a distribution. Support of a distribution

remark suppose $S, T \in \mathcal{D}'(\Omega)$

¿ what does it mean $S = T$?

$\boxed{\forall \varphi \in \mathcal{D}(\Omega), \quad S(\varphi) = T(\varphi)}$ for all the test functions

instead suppose that $\forall x_0 \in \Omega, \ \exists U_{x_0}$ neighborhood of x_0

weaker $\circledast$ s.t. $\forall \psi \in \mathcal{C}_0^\infty(U_{x_0}), \quad S(\psi) = T(\psi)$

only for test functions in a neighborhood of each point

Theorem if $\circledast$ is valid then $T = S$

proof.

consider $\varphi \in \mathcal{D}(\Omega)$

support of φ is compact in Ω

let $x_0 \in \operatorname{supp} \varphi$ then $\exists U_{x_0}$ neighborhood of x_0

such that $\circledast$ is valid

$\forall \psi \in \mathcal{C}_0^\infty(U_{x_0}), \quad T(\psi) = S(\psi)$

now $\{U_{x_0}, x_0 \in \operatorname{supp} \varphi\}$ is covering of $\operatorname{supp} \varphi$

consider $U_1, \ldots U_N$ a finite subcovery

and let $\theta_1, \theta_2, \ldots \theta_N$ a partition of unity for $\operatorname{supp} \varphi$ w.r.t. $U_1, \ldots, U_N$

$\left(\forall j, \quad \theta_j \in \mathcal{C}_0^\infty(U_j) \text{ and } \forall x \in \operatorname{supp} \varphi, \ \sum_{j=1}^N \theta_j(x) = 1 \right)$

then.

$$
\begin{aligned}
T(\varphi) &= T\left(\left(\sum_{j=1}^N \theta_j(x)\right) \varphi(x) \right) \\
&= T\left(\sum_{j=1}^N \left(\theta_j(x) \varphi(x)\right) \right) \\
&= \sum_{j=1}^N T(\theta_j \varphi) \qquad \theta_j \varphi \in \mathcal{C}_0^\infty(U_j) \\
&= \sum_{j=1}^N S(\theta_j \varphi) \\
&= S\left(\sum_{j=1}^N \theta_j \varphi \right) \\
&= S(\varphi)
\end{aligned}
$$

Q.E.D.

def. let Ω open set let $x_0 \in \Omega$

let $T \in \mathcal{D}'(\Omega)$, T is said to be 0 in x_0

if $\exists U_{x_0}$ nbh of x_0 s.t. $T(\varphi) = 0$ $\forall \varphi \in \mathcal{C}_0^\infty(U_{x_0})$

supp T is minimal closed set in Ω outside of which T is identically 0.

Ex. $\operatorname{supp} \delta_0 = \{0\}$ suppose we are in $\mathbb{R}$

take $x > 0$ then $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ with $\operatorname{supp} \varphi \subseteq]x/2, 3/2 x[$

then $\delta_0(\varphi) = 0$

similarly if $x < 0$

Exercise take $f \in \mathcal{C}(\Omega)$

supp f as a continuous function

supp f as a L^1_{loc} function

supp f as a $\mathcal{D}'(\Omega)$

prove that the support is always the same

Derivative of a distribution

idea: given $T \in \mathcal{D}'(\Omega)$ I want to define $S \in \mathcal{D}'(\Omega)$ which acts as the derivative of T

take $f \in \mathcal{C}^1(\Omega)$ $\quad f \longrightarrow T_f$

in distributional sense

define $\partial_{x_j}^{(d)}(T_f)$ in such a way that $\partial_{x_j}^{(d)}(T_f) = T_{\partial_{x_j}^{(cl)} f}$

classical way

$$T_{\partial_{x_j}^{(cl)} f}(\varphi) = \int_\Omega \partial_{x_j}^{(cl)} f(x)\,\varphi(x)\,dx$$

$$= \int_\Omega \left(\partial_{x_j}(f\varphi) - f\cdot\partial_{x_j}\varphi\right)dx$$

$$= \cancel{\underbrace{\int_\Omega \partial_{x_j}(f\varphi)\,dx}_{=\int\dots\int\left(\int \partial_{x_j}(f\varphi)\,dx_j\right)\dots}} - \int_\Omega f\,\partial_{x_j}\varphi\,dx$$

$f\varphi\Big|_*^*$ border (φ has compact support) $= 0$

conclusion

$$T_{\partial_{x_j} f}(\varphi) = \int_\Omega \partial_{x_j} f\cdot\varphi = -\int f\,\partial_{x_j}\varphi = -T_f(\partial_{x_j}\varphi)$$

$$\parallel$$

$$\left(\partial_{x_j}^{(d)}(T_f)\right)(\varphi)$$

def.

$$\boxed{\partial_{x_j}^{(d)} T(\varphi) = -T\left(\partial_{x_j}^{(cl)}\varphi\right)}$$

I define $\partial_{x_j}^{(d)} T$