Chapter 4

The Normalization Lemma

Even if it is known as Normalization "Lemma", this is a deep theorem in algebra, with many applications, not merely a lemma to prove the Nullstellensatz. Later we will see how it is used to study the dimension of K-algebras (Chapter 8) and its interesting geometric interpretation (Theorem 17.1.3).

It takes its name from Emmy Noether, who in 1926 proved it under the hypothesis that K is infinite. The case where K is a finite field was proved by Oscar Zariski in 1943. To prove the Normalization Lemma, we will first see a couple of results about integral elements over a ring. Then we will see a proof over an **infinite field**, rather similar to the original one. It is less technical than any proof of the general case. For other proofs see [AM] or [L].

Let $A \subseteq B$ be rings, where A is a subring of B. In this case we also say that B is an A-algebra. Note that B has a natural structure of A-module. If B is finitely generated **as A-module**, then B is called a **finite A-algebra**. This means that there exist elements $b_1, \dots, b_r \in B$ such that $B = b_1A + b_2A + \ldots + b_rA$, i.e. any element of B is a linear combination with coefficients in A of the generators $b_1, \dots, b_r$: if $b \in B$, then there is an expression $b = a_1b_1 + \cdots + a_rb_r$, with $a_1, \dots, a_r \in A$.

If B is finitely generated **as a ring containing A**, then B is called a **finitely generated A-algebra**. In this case there exists a finite number of elements of B, $b_1, \dots, b_r$, such that $B = A[b_1, \dots, b_r]$, i.e., B is the minimal ring containing A and the elements $b_1, \dots, b_r$. For any element of B there is an expression as polynomial with coefficients in A in the elements $b_1, \dots, b_r$. Another way to express that B is a finitely generated A-algebra is saying that B is (isomorphic to) a quotient of a polynomial ring in a finite number of variables with coefficients in A. Indeed, if $B = A[b_1, \dots, b_r]$, we can define a surjective ring homomorphism φ mapping any polynomial $f(x_1, \dots, x_r) \in A[x_1, \dots, x_r]$ to $f(b_1, \dots, b_r)$. So, by the homomorphism theorem, $B \simeq A[x_1, \ldots, x_r]/\ker\varphi$.

Theorem 4.0.1. *Let $b \in B$, let $A[b] \subseteq B$ be the A-algebra generated by b: $A \subseteq A[b] \subseteq B$. The following are equivalent:*

1. *b is integral over A;*
2. *$A[b]$ is a finite A-algebra;*
3. *there exists a subring C of B, with $A[b] \subseteq C \subseteq B$, such that C is a finite A-algebra.*

Proof. 1) $\Rightarrow$ 2) $A[b]$ is generated by all the powers of b as A-module; we will prove that it is generated by finitely many powers of b. By assumption there is a relation $b^n + a_1 b^{n-1} + \cdots + a_n = 0$, with $a_1, \ldots, a_n \in A$. Therefore, for any $r \geq 0$, $b^{n+r} = -(a_1 b^{n+r-1} + \cdots + a_n b^r)$. By induction on r it follows that all positive powers of b belong to the A-module generated by $1, b, \ldots, b^{n-1}$.

2) $\Rightarrow$ 3) It is enough to take $C = A[b]$.

3) $\Rightarrow$ 1) Let $c_1, \ldots, c_r$ be generators of C as A-module: $C = c_1 A + \cdots + c_r A$. Then, for any $i = 1, \ldots, r$, bc_i is a linear combination of $c_1, \ldots, c_r$ with coefficients in A. So there exists an $r \times r$ matrix $M = (m_{ij})_{i,j=1\ldots,r}$ with entries in A such that

$$bc_i = \sum_{j=1}^{r} m_{ij} c_j, \tag{4.1}$$

i.e. $(bE_r - M)\underline{c} = 0$, where $\underline{c} = {}^t(c_1 \ldots c_r)$ and E_r is the identity matrix. Multiplying both members of equation (4.1) at the left by the adjoint matrix ${}^{ad}(bE_r - M)$, we get $\det(bE_r - M)c_i = 0$ for any i. Since $c_1, \ldots, c_r$ generate C, there is an expression $1 = c_1\alpha_1 + \cdots + c_r\alpha_r$. Therefore $\det(bE_r - M) = \det(bE_r - M)\cdot 1 = \det(bE_r - M)c_1\alpha_1 + \cdots + \det(bE_r - M)c_r\alpha_r = 0$. The expansion of $\det(bE_r - M)$ gives a relation of integral dependence of b over A. □

Corollary 4.0.2. *If $b \in B$ is integral over A, then $A[b]$ is integral extension of A.*

Proof. If $y \in A[b]$, then $A[y] \subset A[b] \subset B$, where $A[b]$ is a finite A-algebra by 2. of Theorem 4.0.1. The conclusion follows from the characterization 3. of integral elements of the same Theorem. □

Remark 5. Equation (4.1) says that b is an eigenvalue of the matrix M. The conclusion is that b is a root of the characteristic polynomial of M. But, since we work over a ring not over a field, we cannot jump straight to the conclusion. In fact we have to use the assumption that $c_1, \ldots, c_r$ generate C as A-module.

Remark 6. We will need also the following easy property, known as "**Transitivity of finiteness**". Let $A \subseteq B$. Suppose that N is a finitely generated B-module. Then N is also an A-module, by restriction of the scalars. Assume also that B is finitely generated as an A-module. Then N is finitely generated as an A-module. Indeed if $y_1, \dots, y_n$ generate N over B and $x_1, \dots, x_m$ generate B as A-module, then the mn products $x_i y_j$ generate N over A.

Corollary 4.0.3. *Let $A \subseteq B$.*

1. *Let $b_1, \dots, b_n \in B$ be integral over A. Then $A[b_1, \dots, b_n]$ is a finite A-module.*

2. **Transitivity of integral dependence**. *Let $A \subset B \subset C$ be rings. If B is integral extension of A and C is integral extension of B, then C is integral extension of A.*

Proof. 1. By induction on n. The case $n = 1$ is part of Theorem 4.0.1. Assume $n > 1$, let $A_r = A[b_1, \dots, b_r]$; then by inductive hypothesis A_{n-1} is a finitely generated A-module. $A_n = A_{n-1}[b_n]$ is a finitely generated A_{n-1}-module by the case $n = 1$, since b_n is integral over A and hence also over A_{n-1}. Then the thesis follows by the transitivity of finiteness (Remark 6).

2. Let $c \in C$, then we have an equation $c^n + b_1 c^{n-1} + \dots + b_n = 0$, with $b_i \in B$ for any index i. The ring $B' = A[b_1, \dots, b_n]$ is a finitely generated A-module by part 1., and $B'[c]$ is a finitely generated B'-module, since c is integral over B'. Hence $B'[c]$ is a finite A-module, by transitivity of finiteness (Remark 6), and therefore c is integral over A by Theorem 4.0.1 3). □

We are now ready to prove

Theorem 4.0.4. Normalization Lemma. *Let $A = K[y_1, \dots, y_n]$ be a finitely generated K–algebra and an integral domain. Let $r :=$ tr.d. $Q(A)/K =$ tr.d. $K(y_1, \dots, y_n)/K$. Then there exist elements $z_1, \dots, z_r \in A$, algebraically independent over K, such that A is integral over the K-algebra $B = K[z_1, \dots, z_r]$.*

Proof. We give a proof by induction on n, assuming that K is infinite.

If $n = 1$, then $A = K[y]$. There are two possibilities, either $r = 1$ or $r = 0$; $r = 1$ if and only if y is transcendental over K, in this case $A = B$; $r = 0$, if and only if y is algebraic over K, in which case A is an algebraic extension of finite degree of K and $B = K$.

Let $n \geq 2$ and assume the theorem is true for K-algebras with $n - 1$ generators. Let $\varphi : K[x_1, \dots, x_n] \to A$ be the surjective homomorphism mapping a polynomial $f(x_1, \dots, x_n)$ to $f(y_1, \dots, y_n)$. If φ is an isomorphism, then $r = n$ and $B = A$. So we assume that

$ker\ \varphi \neq (0)$ and $r < n$: there exists a non-zero polynomial f such that $f(y_1, \ldots, y_n) = 0$. Possibly renaming the variables, we can assume that x_n appears explicitly in f.

If f is monic of degree d with respect to x_n, then $A = K[y_1, \ldots, y_n]$ is a finite module over $K[y_1, \ldots, y_{n-1}]$, generated by $1, y_n, \ldots, y_n^{d-1}$. By Theorem 4.0.1, every element of A is integral over $K[y_1, \ldots, y_{n-1}]$. By inductive assumption, there exists $B = K[z_1, \ldots, z_r]$ with $z_1, \ldots, z_r$ algebraically independent over K, such that $K[y_1, \ldots, y_{n-1}]$ is integral over B. By Transitivity of integral dependence (Corollary 4.0.3 2.), also A is integral over B.

It remains the case where in the kernel of φ there is no monic polynomial in x_n. We claim that we can "change coordinates" **linearly** in $K[x_1, \ldots, x_n]$ in such a way that the polynomial f becomes monic. This means that there is another surjection $K[x_1, \ldots, x_n] \to A$ such that some element of the kernel is monic in x_n.

We consider the linear change of coordinates $x_i \to x_i + a_i x_n$, for $1 \leq i \leq n-1$ and $x_n \to x_n$, where the a_i's are suitable elements of K we are going to choose. Write f as sum of its homogeneous components $f = f_d +$ lower degree terms, where $d = \deg f$. Under this transformation, $f \to f(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1} x_n, x_n)$. We claim it is possible to choose the coefficients a_i so that in this new polynomial the coefficient of x_n^d is non-zero. Just replacing we get $f(x_1+a_1x_n, \ldots, x_{n-1}+a_{n-1}x_n, x_n) = f_d(x_1+a_1x_n, \ldots, x_{n-1}+a_{n-1}x_n, x_n)+$ lower degree terms. Then we expand the top degree term and we get $f_d(x_1+a_1x_n, \ldots, x_{n-1}+a_{n-1}x_n, x_n) = f_d(a_1, \ldots, a_{n-1}, 1)x_n^d +$ lower degree terms in x_n. Adding gives

$$f(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1} x_n, x_n) = f_d(a_1, \ldots, a_{n-1}, 1) x_n^d + \text{lower degree terms in } x_n.$$

Thus we just have to choose the a_i's so that $f_d(a_1, \ldots, a_{n-1}, 1) \neq 0$. Since f_d is a non-zero homogeneous polynomial of degree $d \geq 1$, $f_d(x_1, \ldots, x_{n-1}, 1)$ is a non-zero polynomial of degree less than or equal to d in $x_1, \ldots, x_{n-1}$. Since the field K is infinite, we are done thanks to Exercise 1 in Chapter 2. $\square$

Remarks. This proof has been adapted from MathOverflow, a "question and answer site for professional mathematicians": https://mathoverflow.net/questions/92354/noether-normalization

The same proof can be found in the book [R]. The original article of Emmy Noether is unfortunately in German [N].

A nice article on Normalization Lemma, by Judith Sally, can be found in the book "Emmy Noether in Bryn Mawr", published in the occasion of her 100th birthday ([jS]).

Emmy Noether (1882-1935) is the founder of modern algebra; her story is very interesting and in some aspects symbolic of the difficulties encountered by women mathematicians. As quoted in Wikipedia "In a letter to The New York Times, Albert Einstein wrote:

In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began. In the realm of algebra, in which the most gifted mathematicians have been busy for centuries, she discovered methods which have proved of enormous importance in the development of the present-day younger generation of mathematicians.

On 2 January 1935, a few months before her death, mathematician Norbert Wiener wrote *"Miss Noether is ... the greatest woman mathematician who has ever lived; and the greatest woman scientist of any sort now living, and a scholar at least on the plane of Madame Curie."*

See also http://www.enciclopediadelledonne.it/biografie/emmy-noether/

Chapter 5

The projective closure

5.1 Projective closure and its ideal

In this chapter we will identify the affine space $\mathbb{A}^n$ with the open subet $U_0 \subset \mathbb{P}^n$. As we have seen in Section 2.6, this is possible via the homeomorphisms, inverse each other, $\varphi_0 : U_0 \to \mathbb{A}^n$ and $j_0 : \mathbb{A}^n \to U_0$. Similar considerations hold for any index $i = 0, \dots, n$.

Given an affine variety $X \subset \mathbb{A}^n = U_0 \subset \mathbb{P}^n$, in this way it becomes a subet of $\mathbb{P}^n$ and it makes sense to consider its closure in the Zariski topology of the projective space.

Definition 5.1.1. The **projective closure** of X, $\overline{X}$, is the closure of X in the Zariski topology of $\mathbb{P}^n$.

Since the map φ_0 is a homeomorphism, we have: $\overline{X} \cap \mathbb{A}^n = X$ because X is closed in $\mathbb{A}^n$. The points of $\overline{X} \cap H_0$, where H_0 is the hyperplane at infinity $V_P(x_0)$, are called the "points at infinity" of X in the fixed embedding.

Remark 7. Note that, if K is an infinite field, then the projective closure of $\mathbb{A}^n$ is $\mathbb{P}^n$, i.e. the affine space is dense in the projective space.

Indeed, let F be a homogeneous polynomial of degree d vanishing along $\mathbb{A}^n = U_0$. We can write $F = F_0 x_0^d + F_1 x_0^{d-1} + \cdots + F_d$, where F_i is a homogeneous polynomial of degree i in $x_1, \dots, x_n$ for any i. By assumption, for every $P(a_1, \dots, a_n) \in \mathbb{A}^n$, $P \in V_P(F)$, i.e. $F(1, a_1, \dots, a_n) = 0 = {}^aF(a_1, \dots, a_n)$. So ${}^aF \in I(\mathbb{A}^n)$. We claim that $I(\mathbb{A}^n) = (0)$: if $n = 1$, this follows from the principle of identity of polynomials, because K is infinite. If $n \geq 2$, assume that $F(a_1, ..., a_n) = 0$ for all $(a_1, ..., a_n) \in K^n$ and consider $F(a_1, ..., a_{n-1}, x)$: either it has positive degree in x for some choice of $(a_1, ..., a_n)$, but then it has finitely many zeros against the assumption; or it is constant in x for any choice of $(a_1, \dots, a_n)$, so F belongs to

$K[x_1, ..., x_{n-1}]$ and we can conclude by induction. So the claim is proved. We get therefore that $F_0 = F_1 = \ldots = F_d = 0$ and $F = 0$.

We want to find the relation between the equations of $X \subset \mathbb{A}^n$ and those of its projective closure $\overline{X} \subset \mathbb{P}^n$.

Proposition 5.1.2. *Let $X \subset \mathbb{A}^n$ be an affine variety, $\overline{X}$ be its projective closure. Then*

$$I_h(\overline{X}) = {}^hI(X) := \langle {}^hF | F \in I(X) \rangle.$$

Proof. Let $F \in I_h(\overline{X})$ be a homogeneous polynomial. If $P(a_1, \ldots, a_n) \in X$, then $[1, a_1, \ldots, a_n] \in \overline{X}$, so $F(1, a_1, \ldots, a_n) = 0 = {}^aF(a_1, \ldots, a_n)$. Hence ${}^aF \in I(X)$. There exists $k \geq 0$ such that $F = (x_0^k)^h({}^aF)$ (see proof of Proposition 2.6.1), so $F \in {}^hI(X)$. Hence $I_h(\overline{X}) \subset {}^hI(X)$.

Conversely, if $G \in I(X)$ and $P(a_1, \ldots, a_n) \in X$, then $G(a_1, \ldots, a_n) = 0 = {}^hG(1, a_1, \ldots, a_n)$, so ${}^hG \in I_h(X)$ (here X is seen as a subset of $\mathbb{P}^n$). So ${}^hI(X) \subset I_h(X)$. Since $I_h(X) = I_h(\overline{X})$ (see Exercise 1), we have the claim. □

In particular, if X is a hypersurface and $I(X) = \langle F \rangle$, then $I_h(\overline{X}) = \langle {}^hF \rangle$.

Next example, that will occupy the rest of this Chapter, will show that, **in general**, from $I(X) = \langle F_1, \ldots, F_r \rangle$, it does not follow ${}^hI(X) = \langle {}^hF_1, \ldots, {}^hF_r \rangle$. Only in the last thirty years, thanks to the development of symbolic algebra and in particular of the theory of Gröbner bases, the problem of characterizing the systems of generators of $I(X)$, whose homogeneization generates ${}^hI(X)$, has been solved.

5.2 An extended example: the skew cubic

The example of the skew cubic is of fundamental importance in algebraic geometry, because of the many geometrical phenomena that appear, and are developed in different classes of varieties of which the skew cubic is the first case.

Example 5.2.1 (The skew cubic)**.** In this example we assume that K is infinite. The affine skew cubic is the following closed subset X of $\mathbb{A}^3$: $X = V(y - x^2, z - x^3)$ (we use variables x, y, z). X is the image of the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ such that $\varphi(t) = (t, t^2, t^3)$. Note that $\varphi : \mathbb{A}^1 \to X$ is a homeomorphism (see Exercise 3, Chapter 1). Let α be the ideal $\langle y - x^2, z - x^3 \rangle$. Note that $X = V(\alpha)$. We claim that $\alpha = I(X) = \{F \in K[x, y, z] \mid F(x, x^2, x^3) = 0 \text{ for any } x \in K\}$. Proceeding as in Chapter 3, Example 3.1.2, we consider the development of any polynomial $G \in K[x, y, z]$ in Taylor series around (x, x^2, x^3), and we get the claim. We observe also that α is a prime ideal; to see this, we consider the ring homomorphism $K[x,y,z] \to K[x]$ such that $F(x,y,z) \to F(x,x^2,x^3)$: it is surjective and its kernel is α, therefore the quotient ring $K[x,y,z]/\alpha$ is isomorphic to $K[x]$, which is an integral domain. Therefore α is prime.

Let $\overline{X}$ be the projective closure of X in $\mathbb{P}^3$. First we will study $\overline{X}$ geometrically, then we will determine its homogeneous ideal. We claim that it is the image of the map $\psi : \mathbb{P}^1 \to \mathbb{P}^3$ such that $\psi([\lambda,\mu]) = [\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$. We identify $\mathbb{A}^1$ with the open subset of $\mathbb{P}^1$ defined by $\lambda \neq 0$ i.e. U_0, and $\mathbb{A}^3$ with the open subset of $\mathbb{P}^3$ defined by $x_0 \neq 0$ (U_0 again). Note that $\psi|_{\mathbb{A}^1} = \varphi$, because $\psi([1,t]) = [1,t,t^2,t^3] =$ via the identification of $\mathbb{A}^3$ with $U_0 = (t,t^2,t^3) = \varphi(t)$. Moreover $\psi([0,1]) = [0,0,0,1]$. So $\psi(\mathbb{P}^1) = X \cup \{[0,0,0,1]\}$.

Let G be a homogeneous polynomial of $K[x_0,x_1,x_2,x_3]$ such that $X \subset V_P(G)$. Then $G(1,t,t^2,t^3) = 0 \ \forall t \in K$, so $G(\lambda^3,\lambda^2\mu,\lambda\mu^2,\mu^3) = 0 \ \forall \mu \in K$, $\forall \lambda \in K^*$. Since K is infinite, then $G(\lambda^3,\lambda^2\mu,\lambda\mu^2,\mu^3)$ is the zero polynomial in λ and μ, so $G(0,0,0,1) = 0$ and $V_P(G) \supset \psi(\mathbb{P}^1)$, therefore $\overline{X} \supset \psi(\mathbb{P}^1)$.

Conversely, we prove that $\psi(\mathbb{P}^1)$ is Zariski closed, more precisely

$$\psi(\mathbb{P}^1) = V_P(F_0,F_1,F_2) \text{ where } F_0 := x_1x_3 - x_2^2, F_1 := x_1x_2 - x_0x_3, F_2 := x_0x_2 - x_1^2.$$

One inclusion is clear: every point of $\mathbb{P}^3$ of coordinates $[\lambda^3,\lambda^2\mu,\lambda\mu^2,\mu^3]$ satisfies the three quadratic equations $F_0 = F_1 = F_2 = 0$. Conversely, let $F_i(y_0,\dots,y_3) = 0 \ \forall i = 1,\dots,3$, i.e. $y_1y_3 = y_2^2$, $y_1y_2 = y_0y_3$, $y_0y_2 = y_1^2$. We observe that either $y_0 \neq 0$ or $y_3 \neq 0$, otherwise also $y_1 = y_2 = 0$.

Assume $y_0 \neq 0$, then, using the three equations, we get

$$[y_0,y_1,y_2,y_3] = [y_0^3, y_0^2y_1, y_0^2y_2, y_0^2y_3] = [y_0^3, y_0^2y_1, y_0y_1^2, y_0y_1y_2] = [y_0^3, y_0^2y_1, y_0y_1^2, y_1^3] = \psi([y_0,y_1]).$$

Similarly, if $y_3 \neq 0$, $[y_0,y_1,y_2,y_3] = \psi([y_2,y_3])$. So $\psi(\mathbb{P}^1) = \overline{X}$.

The three polynomials F_0,F_1,F_2 are the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

with entries in $K[x_0,x_1,x_2,x_3]$. Let $F = y - x^2$, $G = z - x^3$ be the two generators of $I(X)$; ${}^hF = x_0x_2 - x_1^2$, ${}^hG = x_0^2x_3 - x_1^3$, hence $V_P({}^hF, {}^hG) = V_P(x_0x_2 - x_1^2, x_0^2x_3 - x_1^3) \neq \overline{X}$, because $V_P({}^hF, {}^hG)$ contains the whole line "at infinity" $V_P(x_0,x_1)$, which is not contained in $\overline{X}$.

We have seen that the projective closure of the affine skew cubic X is $\overline{X} = V_P(F_0,F_1,F_2)$; we shall prove now the non-trivial fact:

Proposition 5.2.2. $I_h(\overline{X}) = \langle F_0, F_1, F_2 \rangle$.

Proof. For any integer number $d \geq 0$, let $I_h(\overline{X})_d := I_h(\overline{X}) \cap K[x_0, x_1, x_2, x_3]_d$: it is a K-vector space of dimension $\leq \binom{d+3}{3}$. We define a K-linear map ρ_d having $I_h(\overline{X})_d$ as kernel:

$$\rho_d : K[x_0, x_1, x_2, x_3]_d \to K[\lambda, \mu]_{3d}$$

such that $\rho_d(F) = F(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3)$. Since ρ_d is clearly surjective, we compute

$$\dim I_h(\overline{X})_d = \binom{d+3}{3} - (3d+1) = (d^3 + 6d^2 - 7d)/6.$$

For $d \geq 2$, we define now a second K-linear map

$$\varphi_d : K[x_0, x_1, x_2, x_3]_{d-2}^{\oplus 3} \to I_h(\overline{X})_d$$

such that $\varphi_d(G_0, G_1, G_2) = G_0F_0 + G_1F_1 + G_2F_2$. Our aim is to prove that φ_d is surjective. The elements of its kernel are called the *syzygies of degree* d among the polynomials F_0, F_1, F_2. Two obvious syzygies of degree 3 are constructed by developing, according to the Laplace rule, the determinant of the matrix obtained repeating one of the rows of M, for example

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

It gives $x_0F_0 + x_1F_1 + x_2F_2 = 0$, so (x_0, x_1, x_2) is a syzygy of degree 3. Similarly (x_1, x_2, x_3).

We put $H_1 = (x_0, x_1, x_2)$ and $H_2 = (x_1, x_2, x_3)$, they both belong to $\ker \varphi_3$. Note that H_1 and H_2 give rise to syzygies of all degrees ≥ 3, in fact we can construct a third linear map

$$\psi_d : K[x_0, x_1, x_2, x_3]_{d-3}^{\oplus 2} \to \ker \varphi_d$$

putting $\psi_d(A, B) = H_1A + H_2B = (x_0, x_1, x_2)A + (x_1, x_2, x_3)B = (x_0A + x_1B, x_1A + x_2B, x_2A + x_3B)$.

Claim. ψ_d is an isomorphism.

Assuming the claim, we are able to compute $\dim \ker \varphi_d = 2\binom{d}{3}$, therefore

$$\dim Im\ \varphi_d = 3\binom{d+1}{3} - 2\binom{d}{3}$$

which coincides with the dimension of $I_h(\overline{X})_d$ previously computed. This proves that φ_d is surjective for all d and concludes the proof of the Proposition.

Proof of the Claim. Let (G_0, G_1, G_2) belong to $\ker \varphi_d$. This means that the following matrix N with entries in $K[x_0, x_1, x_2, x_3]$ is non-invertible:

$$N := \begin{pmatrix} G_0 & G_1 & G_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Therefore, the rows of N are linearly dependent over the quotient field of the polynomial ring $K(x_0, \ldots, x_3)$. Since the last two rows are linearly independent, there exist reduced rational functions $\frac{a_1}{a_0}, \frac{b_1}{b_0} \in K(x_0, x_1, x_2, x_3)$, such that

$$G_0 = \frac{a_1}{a_0}x_0 + \frac{b_1}{b_0}x_1 = \frac{a_1b_0x_0 + a_0b_1x_1}{a_0b_0}$$

and similarly

$$G_1 = \frac{a_1b_0x_1 + a_0b_1x_2}{a_0b_0}, G_2 = \frac{a_1b_0x_2 + a_0b_1x_3}{a_0b_0}$$

The G_i's are polynomials, therefore the denominator a_0b_0 divides the numerator in each of the three expressions on the right hand side. Moreover, if p is a prime factor of a_0, then p divides the three products b_0x_0, b_0x_1, b_0x_2, hence p divides b_0. We can repeat the reasoning for a prime divisor of b_0, so obtaining that $a_0 = b_0$ (up to invertible constants). We get:

$$G_0 = \frac{a_1x_0 + b_1x_1}{b_0}, G_1 = \frac{a_1x_1 + b_1x_2}{b_0}, G_2 = \frac{a_1x_2 + b_1x_3}{b_0},$$

therefore b_0 divides the numerators

$$c_0 := a_1x_0 + b_1x_1, c_1 := a_1x_1 + b_1x_2, c_2 := a_1x_2 + b_1x_3.$$

Hence b_0 divides also $x_1c_0 - x_0c_1 = b_1(x_1^2 - x_0x_1) = -b_1F_2$, and similarly $x_2c_0 - x_0c_2 = b_1F_1$, $x_2c_1 - x_1c_2 = -b_1F_0$. But F_0, F_1, F_2 are irreducible and coprime, so we conclude that $b_0 \mid b_1$. But b_0 and b_1 are coprime, so finally we get $b_0 = a_0 = 1$. $\square$

As an important by-product of the proof of Proposition 5.2.2 we have the **minimal free resolution** of the R-module $I_h(\overline{X})$, where $R = K[x_0, x_1, x_2, x_3]$:

$$0 \to R^{\oplus 2} \xrightarrow{\psi} R^{\oplus 3} \xrightarrow{\varphi} I_h(\overline{X}) \to 0$$

where ψ is represented by the transposed of the matrix M and φ by the triple of polynomials (F_0, F_1, F_2).

Exercises 5.2.3.

1. Let $X \subset \mathbb{A}^n$ be a closed subset, $\overline{X}$ be its projective closure in $\mathbb{P}^n$. Prove that $I_h(X) = I_h(\overline{X})$.

2. Find a system of generators of the ideal of the affine skew cubic X, such that, if you homogeneize them, you get a system of generators for $I_h(\overline{X})$.

Chapter 6

Irreducible components

6.1 Irreducible topological spaces

The aim of this chapter is to introduce the irreducible components of the affine varieties, the "building blocks" of the algebraic varieties. The idea is that the irreducible varieties are a generalization in any dimensions of the irreducible hypersurfaces: any hypersurface is a finite union of irreducible hypersurfaces, similarly any algebraic variety (affine or projective) is a finite unione of irreducible varieties. The notion of irreducible topological space is typical of algebraic geometry and is interesting in this context, although it is not so for Hausdorff topological spaces.

Definition 6.1.1. Let X be a topological space. X is *irreducible* if it is not empty and the following condition holds: if $X = X_1 \cup X_2$ with X_1, X_2 closed subsets of X, then either $X = X_1$ or $X = X_2$.

Equivalently, passing to the complementar sets, X is irreducible if it is non empty and, for all pair of non–empty open subsets U, V, we have $U \cap V \neq \emptyset$.

Note that, by definition, $\emptyset$ is not irreducible.

Proposition 6.1.2. *X is irreducible if and only if any non–empty open subset U of X is dense in X.*

Proof. Let X be irreducible, let P be a point of X and let I_P be an open neighbourhood of P in X. I_P and U are non–empty and open, so $I_P \cap U \neq \emptyset$, therefore $P \in \overline{U}$. This proves that $\overline{U} = X$.

Conversely, assume that all open subsets are dense. Let U, $V \neq \emptyset$ be open subsets. Let $P \in U$ be a point. By assumption $P \in \overline{V} = X$, so $V \cap U \neq \emptyset$ (U is an open neighbourhood of P). $\square$

Example 6.1.3. *1. If $X = \{P\}$ is a unique point, then X is irreducible.*

2. Let K be an infinite field. Then $\mathbb{A}^1$ is irreducible, because proper closed subsets are finite sets. The same holds for $\mathbb{P}^1$.

3. Let $f : X \to Y$ be a continuous map of topological spaces. If X is irreducible and f is surjective, then Y is irreducible.

4. Let $Y \subset X$, $Y \neq \emptyset$, be a subset endowed with the induced topology. Then Y is irreducible if and only if the following holds: if $Y \subset Z_1 \cup Z_2$, with Z_1 and Z_2 closed in X, then either $Y \subset Z_1$ or $Y \subset Z_2$; equivalently: if $Y \cap U \neq \emptyset$, $Y \cap V \neq \emptyset$, with U, V open subsets of X, then $Y \cap U \cap V \neq \emptyset$.

Proposition 6.1.4. *Let X be a topological space, Y a subset of X. Y is irreducible if and only if $\overline{Y}$ is irreducible.*

Proof. Note first that if $U \subset X$ is open and $U \cap Y = \emptyset$ then $U \cap \overline{Y} = \emptyset$. Otherwise, if $P \in U \cap \overline{Y}$, let A be an open neighbourhood of P: then $A \cap Y \neq \emptyset$. In particular, U is an open neighbourhood of P so $U \cap Y \neq \emptyset$.

Let Y be irreducible. If U and V are open subsets of X such that $U \cap \overline{Y} \neq \emptyset$, $V \cap \overline{Y} \neq \emptyset$, then $U \cap Y \neq \emptyset$ and $V \cap Y \neq \emptyset$ so $Y \cap U \cap V \neq \emptyset$ by the irreducibility of Y. Hence $\overline{Y} \cap (U \cap V) \neq \emptyset$. So $\overline{Y}$ is irreducible. If $\overline{Y}$ is irreducible, we get the irreducibility of Y in a completely analogous way. $\square$

Corollary 6.1.5. *Let X be an irreducible topological space and let U be a non–empty open subset of X. Then U is irreducible.*

Proof. By Proposition 6.1.2 $\overline{U} = X$, which is irreducible. By Proposition 6.1.4 U is irreducible. $\square$

6.2 Irreducible algebraic varieties

For algebraic sets (both affine and projective) irreducibility can be expressed in a purely algebraic way.

Proposition 6.2.1. *Let $X \subset \mathbb{A}^n$ (resp. $\mathbb{P}^n$) be an algebraic variety equipped with the Zariski topology, i.e. the induced topology by the Zariski topology of the affine (or projective) space. X is irreducible if and only if $I(X)$ (resp. $I_h(X)$) is prime.*

Proof. Assume first that X is irreducible, $X \subset \mathbb{A}^n$. Let F, G be polynomials in $K[x_1, \ldots, x_n]$ such that $FG \in I(X)$: then

$$V(F) \cup V(G) = V(FG) \supset V(I(X)) = X,$$

hence either $X \subset V(F)$ or $X \subset V(G)$. In the former case, if $P \in X$ then $F(P) = 0$, so $F \in I(X)$, in the second case $G \in I(X)$; hence $I(X)$ is prime.

Assume now that $I(X)$ is prime. Let $X = X_1 \cup X_2$ be the union of two closed subsets. Then $I(X) = I(X_1) \cap I(X_2)$ (see Lesson 4). Assume that $X_1 \neq X$, then $I(X_1)$ strictly contains $I(X)$, otherwise, if $I(X) = I(X_1)$, it would follow $X_1 = V(I(X_1)) = V(I(X)) = X$ because both are closed. So there exists $F \in I(X_1)$ such that $F \notin I(X)$. But for every $G \in I(X_2)$, $FG \in I(X_1) \cap I(X_2) = I(X)$, which is prime: since $F \notin I(X)$, then $G \in I(X)$. So $I(X_2) \subset I(X)$, and we conclude that $I(X_2) = I(X)$, so $X_2 = X$.

If $X \subset \mathbb{P}^n$, the proof is similar, taking into account the following Lemma.

Lemma 6.2.2. *Let $\mathcal{P} \subset K[x_0, x_1, \ldots, x_n]$ be a homogeneous ideal. Then $\mathcal{P}$ is prime if and only if, for every pair of homogeneous polynomials F, G such that $FG \in \mathcal{P}$, either $F \in \mathcal{P}$ or $G \in \mathcal{P}$.*

Proof of the Lemma. Let H, K be any polynomials such that $HK \in \mathcal{P}$. Let $H = H_0 + H_1 + \cdots + H_d$, $K = K_0 + K_1 + \cdots + K_e$ (with $H_d \neq 0 \neq K_e$) be their expressions as sums of homogeneous polynomials. Then $HK = H_0K_0 + (H_0K_1 + H_1K_0) + \cdots + H_dK_e$: the last product is the homogeneous component of degree $d + e$ of HK. $\mathcal{P}$ being homogeneous, $H_dK_e \in \mathcal{P}$; by assumption either $H_d \in \mathcal{P}$ or $K_e \in \mathcal{P}$. In the former case, $HK - H_dK = (H - H_d)K$ belongs to $\mathcal{P}$ while in the second one $H(K - K_e) \in \mathcal{P}$. So in both cases we can proceed by induction. $\square$

We list now some consequences of Proposition 6.2.1.

1. Let K be an infinite field. Then $\mathbb{A}^n$ and $\mathbb{P}^n$ are irreducible, because $I(\mathbb{A}^n) = I_h(\mathbb{P}^n) = (0)$.

2. Let $Y \subset \mathbb{P}^n$ be closed. Y is irreducible if and only if its affine cone $C(Y)$ is irreducible.

3. Let $Y = V(F) \subset \mathbb{A}^n$, be a hypersurface over an algebraically closed field K. If F is irreducible, then Y is irreducible.

4. Let K be algebraically closed. There is a bijection between prime ideals of $K[x_1, \ldots, x_n]$ and irreducible algebraic subsets of $\mathbb{A}^n$. In particular, the maximal ideals correspond to the

points. Similarly, there is a bijection between homogeneous non–irrelevant prime ideals of $K[x_0, x_1, \dots, x_n]$ and irreducible algebraic subsets of $\mathbb{P}^n$.

6.3 Irreducible components

Our next task is to prove that any algebraic variety can be written as a **finite** union of irreducible varieties.

Definition 6.3.1. A topological space X is called *noetherian* if it satisfies the following equivalent conditions:

(i) the ascending chain condition for open subsets;

(ii) the descending chain condition for closed subsets;

(iii) any non–empty set of open subsets of X has maximal elements;

(iv) any non–empty set of closed subsets of X has minimal elements.

The proof of the equivalence is standard (compare with the properties defining noetherian rings).

Example 6.3.2. *$\mathbb{A}^n$ is noetherian: if the following is a descending chain of closed subsets of $\mathbb{A}^n$*

$$Y_1 \supset Y_2 \supset \cdots \supset Y_k \supset \dots,$$

then

$$I(Y_1) \subset I(Y_2) \subset \cdots \subset I(Y_k) \subset \dots$$

is an ascending chain of ideals of $K[x_1, \dots, x_n]$, hence it is stationary from a suitable m on; therefore $V(I(Y_m)) = Y_m = V(I(Y_{m+1})) = Y_{m+1} = \dots$.

Similarly $\mathbb{P}^n$ is noetherian.

Proposition 6.3.3. *Let X be a noetherian topological space and Y be a non–empty closed subset of X. Then Y can be written as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets. The maximal Y_i's in the union are uniquely determined by Y and are called the "irreducible components" of Y. They are the maximal irreducible subsets of Y.*

Proof. By contradiction. Let S be the set of the non–empty closed subsets of X which are not a finite union of irreducible closed subsets: assume $S \neq \emptyset$. By noetherianity S has minimal elements, fix one of them Z. Z is not irreducible, so $Z = Z_1 \cup Z_2$, $Z_i \neq Z$ for $i = 1, 2$. So $Z_1, Z_2 \notin S$, hence Z_1, Z_2 are both finite unions of irreducible closed subsets, so such is Z: a contradiction.

Now assume that $Y = Y_1 \cup \cdots \cup Y_r$, with $Y_i \not\subseteq Y_j$ if $i \neq j$ and Y_i irreducible closed for all i. If there is another similar expression $Y = Y'_1 \cup \cdots \cup Y'_s$, $Y'_i \not\subseteq Y'_j$ for $i \neq j$, then $Y'_1 \subset Y_1 \cup \ldots Y_r$, so $Y'_1 = \bigcup_{i=1}^{r}(Y'_1 \cap Y_i)$, hence $Y'_1 \subset Y_i$ for some i, and we can assume $i = 1$. Similarly, $Y_1 \subset Y'_j$, for some j, so $Y'_1 \subset Y_1 \subset Y'_j$, so $j = 1$ and $Y_1 = Y'_1$. Now let $Z = \overline{Y - Y_1} = Y_2 \cup \cdots \cup Y_r = Y'_2 \cup \cdots \cup Y'_s$ and proceed by induction. $\square$

Corollary 6.3.4. *Any algebraic variety in $\mathbb{A}^n$ (resp. in $\mathbb{P}^n$) can be written in a unique way as the finite union of its irreducible components.*

Note that the irreducible components of X are its maximal irreducible algebraic subsets. They correspond to the minimal prime ideals over $I(X)$. Since $I(X)$ is radical, these minimal prime ideals coincide with the primary ideals appearing in the primary decomposition of $I(X)$.

6.4 Quasi–projective varieties

Often the irreducible closed subsets of $\mathbb{A}^n$ are called *affine varieties*, i.e., the term variety is reserved to the irreducible ones. Similarly for the irreducible closed subsets of $\mathbb{P}^n$.

Definition 6.4.1. A locally closed subset in $\mathbb{P}^n$ is the intersection of an open and a closed subset. An irreducible locally closed subset of $\mathbb{P}^n$ is called a *quasi–projective variety:* it is open in an irreducible closed subset Z of $\mathbb{P}^n$, therefore it is dense in Z.

We conclude this chapter with the (non-trivial) proof of the irreducibility of the product of irreducible affine varieties.

Proposition 6.4.2. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be irreducible affine varieties. Then $X \times Y$ is an irreducible subvariety of $\mathbb{A}^{n+m}$.*

Proof. Let $X \times Y = W_1 \cup W_2$, with W_1, W_2 closed. For any $P \in X$, the map $\{P\} \times Y \to Y$ which takes (P, Q) to Q is a homeomorphism, so $\{P\} \times Y$ is irreducible. $\{P\} \times Y = (W_1 \cap (\{P\} \times Y)) \cup (W_2 \cap (\{P\} \times Y))$, so $\exists i \in \{1, 2\}$ such that $\{P\} \times Y \subset W_i$. Let $X_i = \{P \in X \mid \{P\} \times Y \subset W_i\}$, $i = 1, 2$. Note that $X = X_1 \cup X_2$.

Claim. X_i is closed in X.
Let $X^i(Q) = \{P \in X \mid (P, Q) \in W_i\}$, $Q \in Y$. We have: $(X \times \{Q\}) \cap W_i = X^i(Q) \times \{Q\} \simeq X^i(Q)$; $X \times \{Q\}$ and W_i are closed in $X \times Y$, so $X^i(Q) \times \{Q\}$ is closed in $X \times Y$ and also in $X \times \{Q\}$, so $X^i(Q)$ is closed in X. Note that $X_i = \bigcap_{Q \in Y} X^i(Q)$, hence X_i is closed, which proves the Claim.

Since X is irreducible, $X = X_1 \cup X_2$ implies that either $X = X_1$ or $X = X_2$, so either $X \times Y = W_1$ or $X \times Y = W_2$. $\square$

Exercises 6.4.3.

1. Let $X \neq \emptyset$ be a topological space. Prove that X is irreducible if and only if all non–empty open subsets of X are connected.

2. Prove that the *cuspidal cubic* $Y \subset \mathbb{A}^2_{\mathbb{C}}$ of equation $x^3 - y^2 = 0$ is irreducible. (Hint: express Y as image of $\mathbb{A}^1$ in a continuous map...)

3. Give an example of two irreducible subvarieties of $\mathbb{P}^3$ whose intersection is reducible.

4. Find the irreducible components of the following algebraic sets over the complex field:

   a) $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subset \mathbb{A}^2$;

   b) $V(y^2 - xz, z^2 - y^3) \subset \mathbb{A}^3$.

5. Let Z be a topological space and let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of Z such that $U_\alpha \cap U_\beta \neq \emptyset$ for $\alpha \neq \beta$ and that all U_α's are irreducible. Prove that Z is irreducible.

Chapter 7

Dimension

7.1 Topological dimension

There are a few equivalent ways to give the definition of dimension for algebraic varieties. In this section we will first see a topological definition, then an algebraic characterization. In a later lesson, we will see a more geometrical interpretation.

Let X be a topological space.

Definition 7.1.1. The *topological dimension* of X is the supremum of the lengths of the chains of distinct irreducible closed subsets of X, where by definiton the following chain has length n:

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n.$$

The topological dimension of X is denoted by $\dim X$. It is also called combinatorial or Krull dimension.

Example 7.1.2.

1. $\dim \mathbb{A}^1 = 1$: the maximal length chains of irreducible closed subsets all have the form $\{P\} \subset \mathbb{A}^1$.

2. $\dim \mathbb{A}^n$: a chain of length n is

$$\{0\} = V(x_1, \ldots, x_n) \subset V(x_1, \ldots, x_{n-1}) \subset \cdots \subset V(x_1) \subset \mathbb{A}^n.$$

Note that $V(x_1, \ldots, x_i)$ is irreducible for any $i \leq n$, because the ideal $\langle x_1, \ldots, x_i \rangle$ is prime. Indeed $K[x_1, \ldots, x_n]/\langle x_1, \ldots, x_i \rangle \simeq K[x_{i+1}, \ldots, x_n]$, which is an integral domain. Therefore we get that $\dim \mathbb{A}^n \geq n$. We will see shortly that proving equality is non trivial. We note also that, from every chain of irreducible closed subsets of $\mathbb{A}^n$, passing to their ideals, we get a chain of the same length of prime ideals in $K[x_1, \ldots, x_n]$.

3. Let X be irreducible. Then $\dim X = 0$ if and only if X is the closure of every point of it.

We prove now some useful relations between the dimensions of X and of its subspaces.

Proposition 7.1.3. *1. If $Y \subset X$ is a subspace of the topological space X with the induced topology, then $\dim Y \leq \dim X$. In particular, if $\dim X$ is finite, then also $\dim Y$ is finite. In this case, the number $\dim X - \dim Y$ is called the* **codimension** *of Y in X.*

2. If $X = \bigcup_{i \in I} U_i$ is an open covering, then $\dim X = \sup_i \{\dim U_i\}$.

3. If X is noetherian and $X_1, \dots, X_s$ are its irreducible components, then $\dim X = \sup_i \dim X_i$.

4. If $Y \subset X$ is closed, X is irreducible, $\dim X$ is finite and $\dim X = \dim Y$, then $Y = X$.

Proof. 1. Let $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ be a chain of irreducible closed subsets of Y. Then taking closures we get the following chain of irreducible closed subsets of X: $\overline{Y_0} \subseteq \overline{Y_1} \subseteq \cdots \subseteq \overline{Y_n}$. Note that, for any index i, $\overline{Y_i} \cap Y = Y_i$, because Y_i is closed into Y, so if $\overline{Y_i} = \overline{Y_{i+1}}$, then $Y_i = Y_{i+1}$. Therefore the two chains have the same length and we can conclude that $\dim Y \leq \dim X$.

2. Let $X_0 \subset X_1 \subset \cdots \subset X_n$ be a chain of irreducible closed subsets of X. Let $P \in X_0$ be a point: there exists an index $i \in I$ such that $P \in U_i$. So $\forall k = 0, \dots, n$ $X_k \cap U_i \neq \emptyset$: it is an irreducible closed subset of U_i, irreducible because open in X_k which is irreducible. Consider

$$X_0 \cap U_i \subset X_1 \cap U_i \subset \cdots \subset X_n \cap U_i;$$

it is a chain of length n, because $\overline{X_k \cap U_i} = X_k$: in fact $X_k \cap U_i$ is open in X_k hence dense. Therefore, for any chain of irreducible closed subsets of X, there exists a chain of the same length of irreducible closed subsets of some U_i. So $\dim X \leq \sup \dim U_i$. By 1., equality holds.

3. Any chain of irreducible closed subsets of X is completely contained in an irreducible component of X. The conclusion follows as in 2.

4. If $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ is a chain of irreducible closed subsets of Y of maximal length, then it is also a maximal length chain in X, because $\dim X = \dim Y$. Hence $X = Y_n$, because X is irreducible, and we conclude that $X \subset Y$. □

Corollary 7.1.4. *$\dim \mathbb{P}^n = \dim \mathbb{A}^n$.*

Proof. The equality follows from $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$, and the homeomorphism of U_i with $\mathbb{A}^n$ for all i. □

If X is noetherian and all its irreducible components have the same dimension r, then X is said to have *pure dimension* r. Note that the topological dimension is invariant by homeomorphism. By definition, a *curve* is an algebraic set of pure dimension 1; a *surface* is an algebraic set of pure dimension 2.

7.2 Dimension of algebraic varieties

We want to study the dimensions of affine algebraic sets. The following definition results to be very important.

Definition 7.2.1. Let $X \subset \mathbb{A}^n$ be an algebraic set. The *coordinate ring* of X is

$$K[X] := K[x_1, \dots, x_n]/I(X).$$

It is a finitely generated reduced K–algebra, i.e. there are no non–zero nilpotents, because $I(X)$ is radical (see Exercise 3, Chapter 3).

There is the canonical epimorphism $K[x_1, \dots, x_n] \to K[X]$ such that $F \to [F]$. The elements of $K[X]$ can be interpreted as *polynomial functions* on X: to a polynomial F, we can associate the function $f : X \to K$ such that $P(a_1, \dots, a_n) \to F(a_1, \dots, a_n)$.

Two polynomials F, G define the same function on X if, and only if, $F(P) = G(P)$ for every point $P \in X$, i.e. if $F - G \in I(X)$, which means exactly that F and G have the same image in $K[X]$.

$K[X]$ is generated as K–algebra by $[x_1], \dots, [x_n]$: they can be interpreted as *coordinate functions* on X. We will denote them by $t_1, \dots, t_n$. In fact $t_i : X \to K$ is the function which associates to $P(a_1, \dots, a_n)$ the coordinate a_i. Note that the function f can be interpreted as $F(t_1, \dots, t_n)$: the polynomial F evaluated at the n– tuple of the coordinate functions.

In the projective space we can do an analogous construction. If $Y \subset \mathbb{P}^n$ is closed, then by definition the *homogeneous coordinate ring* of Y is

$$S(Y) := K[x_0, x_1, \dots, x_n]/I_h(Y).$$

Also $S(Y)$ is a finitely generated reduced K–algebra, but its elements cannot be interpreted as functions on Y. They are functions on the cone $C(Y)$.

We note that, from the fact that $I_h(Y)$ is homogeneous it follows that also $S(Y)$ is a graded ring, with the graduation induced by the polynomial ring. Indeed, if $F - G \in I_h(Y)$, and $F = F_0 + \ldots + F_d$, $G = G_0 + \ldots + G_e$ are their decompositions in homogeneous components then it follows that $F_0 - G_0 \in I_h(Y)$, $F_1 - G_1 \in I_h(Y)$, and so on. Therefore $S(Y) = \oplus_{d \geq 0} S(Y)_d$, where $S(Y)_d$ is the subgroup of the classes of homogeneous polynomials of degree d.

Definition 7.2.2. Let R be a ring. The *Krull dimension* of R is the supremum of the lengths of the chains of prime ideals of R

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_r.$$

Similarly, the *heigth* of a prime ideal $\mathcal{P}$ is the sup of the lengths of the chains of prime ideals contained in $\mathcal{P}$: it is denoted $\mathrm{ht}\mathcal{P}$.

Proposition 7.2.3. *Let K be an algebraically closed field. Let X be an affine algebraic set contained in $\mathbb{A}^n$. Then $\dim X = \dim K[X]$. In particular $\dim \mathbb{A}^n = \dim K[x_1, \ldots, x_n]$.*

Proof. By the Nullstellensatz and its Corollary 3.2.9 the chains of irreducible closed subsets of X correspond bijectively to the chains of prime ideals of $K[x_1, \ldots, x_n]$ containing $I(X)$, and therefore also to the chains of prime ideals of the quotient ring $K[X] = K[x_1, \ldots, x_n]/I(X)$. $\square$

The dimension theory for commutative rings contains some important theorems about the dimension of K–algebras. The following theorem states the basic properties in the case of integral domains and the algebraic characterization of dimension for affine varieties.

Theorem 7.2.4. *Let K be any field. Let A be a finitely generated K–algebra and an integral domain.*

1. $\dim A = tr.d.Q(A)/K$*, where $Q(A)$ is the quotient field of A. In particular* $\dim A$ *is finite.*
2. *Let $\mathcal{P} \subset A$ be any prime ideal. Then* $\dim A = \mathrm{ht}\mathcal{P} + \dim A/\mathcal{P}$.

Proof. We postpone the proof to next chapter. It relies on the Normalization Lemma and on the Cohen-Seidenberg theorems about the structure of prime ideals for integral extensions of K-algebras. $\square$

Corollary 7.2.5. *Let K be an algebraically closed field.*

1. $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.
2. *If X is an irreducible affine variety, then* $\dim X = tr.d.K(X)/K$*, where $K(X)$ denotes the quotient field of $K[X]$.*

3. If $X \subset \mathbb{A}^n$ is an irreducible affine variety, then $\dim X = n - \mathrm{ht} I(X)$.

Proof. 1. $\dim K[x_1, \dots, x_n] = tr.d.K(x_1, \dots, x_n)/K = n$.

2. follows immediately from Theorem 7.2.4, 1.

3. is Theorem 7.2.4, 2, applied to the case $A = K[x_1, \dots, x_n]$ and $\mathcal{P} = I(X)$. □

Note that the homogeneous coordinate ring of $\mathbb{P}^n$ is $K[x_0, \dots, x_n]$, whose dimension is $n+1$, strictly bigger than the dimension of $\mathbb{P}^n$. Similarly, if Y is a projective algebraic variety, then $\dim S(Y) = \dim C(Y)$, the affine cone over Y.

Corollary 7.2.5 tells us how to compute the dimension of an affine irreducible variety over an algebraically closed field K. If X is a reducible affine variety, and $X = X_1 \cup \dots \cup X_r$ is its decomposition as union of irreducible components, then $\dim X$ is the maximum of the dimensions $\dim X_i$.

The following is the characterization of the algebraic varieties of codimension 1 in $\mathbb{A}^n$.

Proposition 7.2.6. *Let $X \subset \mathbb{A}^n$ be an affine variety over an algebraically closed field. Then X is a hypersurface if and only if X is of pure dimension $n-1$.*

Proof. Let $X \subset \mathbb{A}^n$ be a hypersurface, with $I(X) = (F) = (F_1 \dots F_s)$, where $F_1, \dots, F_s$ are the (distinct) irreducible factors of F all of multiplicity one. Then $X = V(F_1 \dots F_s) = V(F_1) \cup \dots \cup V(F_s)$; therefore $V(F_1)$,..., $V(F_s)$ are the irreducible components of X, whose ideals are (F_1), ..., (F_s). So it is enough to prove that $\mathrm{ht}(F_i) = 1$, for $i = 1, \dots, s$.

If $\mathcal{P} \subset (F_i)$ is a prime ideal, then either $\mathcal{P} = (0)$ or there exists $G \in \mathcal{P}$, $G \neq 0$. In the second case, let A be an irreducible factor of G belonging to $\mathcal{P}$: $A \in (F_i)$ so $A = HF_i$. Since A is irreducible, either H or F_i is invertible; F_i is irreducible, so H is invertible, hence $(A) = (F_i) \subset \mathcal{P}$. Therefore either $\mathcal{P} = (0)$ or $\mathcal{P} = (F_i)$, and $\mathrm{ht}(F_i) = 1$.

Conversely, assume that X is irreducible of dimension $n-1$. Since $X \neq \mathbb{A}^n$, there exists $F \in I(X)$, $F \neq 0$, with irreducible factorization $F = F_1 \dots F_s$. Hence $X \subset V(F) = V(F_1) \cup \dots \cup V(F_s)$. By the irreducibility of X, $X \subset V(F_i)$, which is irreducible of dimension $n-1$, by the first part. So $X = V(F_i)$ (by Proposition 7.1.3, 4). □

This proposition does not generalise to higher codimension. There exist codimension 2 algebraic subsets of $\mathbb{A}^n$ whose ideal is not generated by two polynomials. An example in $\mathbb{A}^3$ is the curve X parametrised by (t^3, t^4, t^5). It is possible to show that a system of generators of $I(X)$ is formed by the three polynomials $x^3 - yz, y^2 - xz, z^2 - x^2y$. One can easily show that $I(X)$ cannot be generated by two polynomials. For a proof and a discussion of this example, and more generally of the ideals of the curves admitting a parametrization of the form $x = t^{n_1}, y = t^{n_2}, z = t^{n_3}$, see [K], Chapter V.

Proposition 7.2.7. *Let $X \subset \mathbb{A}^n_K$, $Y \subset \mathbb{A}^m_K$ be irreducible closed subsets, over an algebraically closed field K. Then* $\dim X \times Y = \dim X + \dim Y$.

Proof. Let $r = \dim X$, $s = \dim Y$; let $t_1, \dots, t_n$ (resp. $u_1, \dots, u_m$) be coordinate functions on $\mathbb{A}^n$ (resp. $\mathbb{A}^m$). We can assume that $t_1, \dots, t_r$ is a transcendence basis of $Q(K[X])$ and $u_1, \dots, u_s$ a transcendence basis of $Q(K[Y])$. By definition, $K[X \times Y]$ is generated as K–algebra by $t_1, \dots, t_n, u_1, \dots, u_m$: we want to show that $t_1, \dots, t_r, u_1, \dots, u_s$ is a transcendence basis of $Q(K[X \times Y])$ over K. Assume that $F(x_1, \dots, x_r, y_1, \dots, y_s)$ is a polynomial which vanishes on $t_1, \dots, t_r, u_1, \dots, u_s$, i.e. F defines the zero function on $X \times Y$. Then, $\forall\ P \in X$, $F(P; y_1, \dots, y_s)$ is zero on Y, i.e. $F(P; u_1, \dots, u_s) = 0$. Since $u_1, \dots, u_s$ are algebraically independent, every coefficient $a_i(P)$ of $F(P; y_1, \dots, y_s)$ is zero, $\forall\ P \in X$. Since $t_1, \dots, t_r$ are algebraically independent, the polynomials $a_i(x_1, \dots, x_r)$ are zero, so $F(x_1, \dots, x_r, y_1, \dots, y_s) = 0$. So $t_1, \dots, t_r, u_1, \dots, u_s$ are algebraically independent. Since this is certainly a maximal algebraically free set, it is a transcendence basis. □

Exercises 7.2.8.

1. Prove that a proper closed subset of an irreducible curve is a finite set. Deduce that any bijection between irreducible curves is a homeomorphism.

2. Let $X \subset \mathbb{A}^2$ be the cuspidal cubic of equation: $x^3 - y^2 = 0$, let $K[X]$ be its coordinate ring. Prove that all elements of $K[X]$ can be written in a unique way in the form $f(x) + yg(x)$, where f, g are polynomials in the variable x. Deduce that $K[X]$ is not isomorphic to a polynomial ring.