MECCANICA RAZIONALE

Ing. Civile & Ambientale
Navale

14 Aprile 2021

Problema della dinamica.

Punto di partenza è il Principio di D'Alembert: ogni problema di dinamica si può considerare come un problema di statica aggiungendo alle forze agenti su ogni punto B le forze $-\frac{d}{dt}\underline{p}_B$ "forze di inerzia"

$$\underline{F}_B = \frac{d}{dt}\underline{p}_B \quad \Rightarrow \quad \underline{F}_B - \frac{d}{dt}\underline{p}_B = \underline{0}$$

forza

Statica — Dinamica

- stazionarietà dell' energia potenziale $dV = 0$ → • eq. di Lagrange conservative

- PLV $LV = \sum_i Q_i \, \delta q_i = 0$ → • eq. di Lagrange non conservative

- Equazioni Cardinali della Statica → • Equazioni Cardinali della Dinamica

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<u>Sistemi olonomi generali</u> a ℓ gradi di libertà

$$\underline{q}(t) = (q_1(t), \dots, q_\ell(t))$$

Mettiamo insieme il Principio dei

Lavori Virtuali & il Principio di D'Alembert

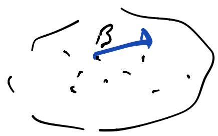
$$\sum_{B\in S}{}' \left(\underline{F}_B - \frac{d}{dt}\underline{p}_B \right)\cdot \delta\underline{x}_B = 0$$

$\forall\ \delta\underline{x}_B$ spostamento virtuale invertibile

[Statico $\displaystyle\sum_{B\in S}{}' \underline{F}_B \cdot \delta\underline{x}_B = 0$]

Ricordiamo: $\displaystyle \delta\underline{x}_B = \sum_{i=1}^{\ell}{}' \frac{\partial \underline{x}_B}{\partial q_i}\,\delta q_i$

$\underline{x}_B = \underline{x}_B(\underline{q})$



$\underline{x}_B \to \underline{x}_B + \delta\underline{x}_B$ segue da $q_i \to q_i + \delta q_i \quad \forall i$

$$\sum_B \left(\underline{F}_B - \frac{d}{dt}\underline{p}_B \right)\delta\underline{x}_B = 0 \Rightarrow$$

$$\sum_{i=1}^{\ell}{}' \left(\sum_{B\in S}{}' \frac{d\underline{p}_B}{dt}\cdot\frac{\partial \underline{x}_B}{\partial q_i} \right)\delta q_i =$$

$$(\dot{p})_i \qquad = \sum_{i=1}^{\ell}{}' \left(\sum_{B\in S} \underline{F}_B \cdot \frac{\partial \underline{x}_B}{\partial q_i} \right)\delta q_i$$

↑ forze generalizzate

$(\dot{P})_i$ derivata della quantità di moto generalizzata
$\left(\text{viene da } \frac{d p_B}{dt} \right)$

↳ $(\dot{P})_i = \underline{Q_i}$

↑

manipolare

Vediamo che

$$(\dot{P})_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i}$$

K energia cinetica

1) $$(\dot{P})_i = \sum_{B \in S} \frac{d p_B}{dt} \frac{\partial x_B}{\partial q_i}$$

$$= \sum_{B \in S}' \left[\frac{d}{dt}\left(p_B \cdot \frac{\partial x_B}{\partial q_i} \right) - p_B \cdot \frac{d}{dt}\left(\frac{\partial x_B}{\partial q_i} \right) \right]$$

$$\frac{d}{d\tau}(f \cdot g) = \left(\frac{d}{d\tau} f\right) g + f\left(\frac{d}{d\tau} g\right)$$

$$= \frac{d}{d\tau}\left(\sum_{B\in S}' \underline{p}_B \cdot \frac{\partial \underline{x}_B}{\partial q_i}\right) - \sum_{B\in S}' \underline{p}_B \cdot \frac{d}{d\tau}\left(\frac{\partial \underline{x}_B}{\partial q_i}\right)$$

2) Consideriamo l' energia cinetica

$$K = \frac{1}{2} \sum_{B\in S}' m_B \|\underline{v}_B\|^2$$

Vediamo come e' fatto $\underline{v}_B$

Per un punto B : $\underline{x}_B = \underline{x}_B(q(\tau), t)$

$$\underline{v}_B = \frac{d}{d\tau} \underline{x}_B(q(\tau), \tau) = \sum_{j=1}^{\ell}{}' \frac{\partial \underline{x}_B}{\partial q_j} \dot{q}_j + \frac{\partial}{\partial \tau} \underline{x}_B$$

in particolare $\frac{\partial}{\partial \dot{q}_i} \underline{v}_B = \frac{\partial}{\partial q_i} \underline{x}_B$

$$\sum_{B\in S} \underline{p}_B \cdot \frac{\partial \underline{x}_B}{\partial q_i} = \sum_{B\in S}' m_B \underline{v}_B \cdot \frac{\partial}{\partial \dot{q}_i} \underline{v}_B$$

$$= \frac{1}{2} \frac{\partial}{\partial \dot q_i} \left(\sum_{B \in S}' m_B \| \underline{v}_B \|^2 \right)$$

$$= \frac{\partial}{\partial \dot q_i} K$$

Cioè abbiamo dimostrato che

$$\sum_{B \in S} \underline{p}_B \frac{\partial \underline{x}_B}{\partial q_i} = \frac{\partial K}{\partial \dot q_i}$$

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3) Guardiamo $\sum_{B \in S}' \underline{p}_B \cdot \underline{\frac{d}{dt}\left(\frac{\partial \underline{x}_B}{\partial q_i}\right)}$

$$\frac{d}{dt}\left(\frac{\partial \underline{x}_B}{\partial q_i}\right) = \sum_{j=1}^{\ell}{}' \frac{\partial}{\partial q_j}\left(\frac{\partial \underline{x}_B}{\partial q_i}\right) \dot q_j + \frac{\partial}{\partial t} \frac{\partial \underline{x}_B}{\partial q_i}$$

$$\underline{x}_B = \underline{x}_B(\underline{q}(t), t)$$

$$\frac{\partial}{\partial q_i} \frac{d}{dt} \underline{x}_B = \frac{\partial}{\partial q_i} \underbrace{\left(\sum_{j=1}^{\ell} \frac{\partial \underline{x}_B}{\partial q_j} \dot q_j + \frac{\partial \underline{x}_B}{\partial t} \right)}_{\underline{v}_B}$$

$$\to \underline{\frac{\partial}{\partial q_i} \underline{v}_B}$$

$$\sum_{B\in S}' \underline{p}_B \cdot \frac{d}{dt}\left(\frac{\partial \underline{x}_B}{\partial q_i}\right) =$$

$$= \sum_{B\in S}' m_B \underline{v}_B \cdot \frac{\partial}{\partial q_i}\underline{v}_B =$$

$$= \frac{\partial}{\partial q_i}\left(\frac{1}{2}\sum_{B\in S}' m_B \|\underline{v}_B\|^2\right)$$

$$= \frac{\partial}{\partial q_i} K$$

Mettiamo tutto insieme

$$(\dot{P})_i = \frac{d}{dt}\left(\sum_{B\in S}\underline{p}_B\cdot\frac{\partial \underline{x}_B}{\partial q_i}\right) - \sum_{B\in S}\underline{p}_B\cdot\frac{d}{dt}\left(\frac{\partial \underline{x}_B}{\partial q_i}\right)$$

$$= \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i}$$

Dal principio di D'Alembert

$$(\dot{P})_i = Q_i$$

Troviamo le <u>equazioni di</u>

Lagrange con conservative

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q_i}\right) - \frac{\partial K}{\partial q_i} = Q_i$$

Caso conservativo : $Q_i = -\frac{\partial V}{\partial q_i}$

Introduciamo la funzione Lagrangiana

$$\mathcal{L} = K - V$$

Allora le eq. di Lagrange diventano

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0}$$

Infatti: $\frac{d}{dt}\frac{\partial}{\partial \dot q_i}[K - V] - \frac{\partial}{\partial q_i}[K - V]$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot q_i} K - \frac{\partial}{\partial q_i} K + \frac{\partial V}{\partial q_i} +$$

$$+ \left(\frac{d}{dt}\frac{\partial}{\partial \dot q_i}(-V)\right) = 0$$

$\hookrightarrow = 0$ perché V non dipende dalle $\dot{q}$ per forze posizionali

$$\rightarrow \quad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} = Q_i$$

quando $Q_i = -\frac{\partial V}{\partial q_i}$

Seconda Parte

$$PLV \xrightarrow{\text{D'Alembert}} \text{equazioni di Lagrange}$$

K energia cinetica ?

$$ECS \longrightarrow ?$$

$$\underline{R}^e - \sum_{B \in S}' \frac{d}{dt} \underline{P}_B = \underline{0}$$

$$\underline{P}_B = m_B \underline{v}_B \rightarrow \underline{P} = \sum_{B \in S}' m_B \underline{v}_B$$

$$\hookrightarrow \quad \underline{R}^e - \frac{d}{dt}\underline{P} = 0$$

dalla definizione di centro di massa

$$\sum_{B\in S} m_B \, \underline{v}_B = M \, \underline{v}_G$$

massa totale

velocità centro di massa

$$\underline{R}^e = \frac{d}{dt}\underline{P} = \frac{d}{dt} M \underline{v}_G$$

Allo stesso modo, per i momenti:

$$\underline{M}^e(O) - \sum_{B\in S}' (\underline{x}_B - \underline{x}_O) \wedge \dot{\underline{p}}_B = \underline{0}$$

$$\sum_{B\in S}' (\underline{x}_B - \underline{x}_O) \wedge \frac{d}{dt}\underline{p}_B =$$

$$= \sum_{B\in S}' \left\{ \underbrace{\frac{d}{dt}\left[(\underline{x}_B - \underline{x}_O)\wedge \underline{p}_B\right]} - \underbrace{\left(\frac{d}{dt}(\underline{x}_B - \underline{x}_O)\right)\wedge \underline{p}_B} \right\}$$

$$\frac{d}{dt}[f \cdot g] = \frac{df}{dt} g + f \frac{dg}{dt}$$

$$\underline{L}(O) = \sum_{B \in S}' (\underline{x}_B - \underline{x}_O) \wedge m_B \underline{v}_B$$

momento angolare del sistema

$$= \frac{d}{dt} \underline{L}(O) - \sum_{B \in S}' (\underline{v}_B - \underline{v}_O) \wedge m_B \underline{v}_B$$

$$= \frac{d}{dt} \underline{L}(O) + \underline{v}_O \wedge \sum_{B \in S}' m_B \underline{v}_B$$

(perché $\underline{v}_B \wedge \underline{v}_B = \underline{0}$)

$$\underline{M}^{\wedge}(O) = \frac{d}{dt} \underline{L}(O) + \underline{v}_O \wedge \underline{P}$$

Ricapitoliamo. Se definiamo

$$\underline{P} = \sum_{B \in S}' m_B \underline{v}_B = M \underline{v}_G$$

quantità di moto del sistema

$$\underline{L}(O) = \sum_{B \in S}' (\underline{x}_B - \underline{x}_O) \wedge m_B \underline{v}_B$$

momento angolare del sistema

Abbiamo le equazioni cardinali della

dinamica

$$\boxed{\begin{aligned}\underline{R}^e &= \frac{d}{dt}\,\underline{P}\\ \underline{M}^e(O) &= \frac{d}{dt}\,\underline{L}(O) + \underline{v}_O \wedge \underline{P}\end{aligned}}$$

Caso particolare in cui $O \equiv G$

$$\underline{M}^e(G) = \frac{d}{dt}\,\underline{L}(G) + \underline{v}_G \wedge \underbrace{\underline{P}}_{M\,\underline{v}_G}$$

$$= \frac{d}{dt}\,\underline{L}(G)$$

Corpo rigido: energia cinetica e momento angolare

Energia cinetica

$$K = \frac{1}{2}\sum_{B\in S}{}' m_B \|\underline{v}_B\|^2$$

$$= \frac{1}{2}\sum_{B\in S}{}' m_B \left\|\underline{v}_O + \underline{\omega}\wedge(\underline{x}_B - \underline{x}_O)\right\|^2$$

per Poisson $\underline{v}_B = \underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)$

$$= \frac{1}{2} \sum_{B \in S}' m_B \left(\underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)\right) \cdot \left(\underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)\right)$$

$$= I_1 + I_2 + I_3$$

$$I_1 = \frac{1}{2} \sum_{B \in R}' m_B \| \underline{v}_0 \|^2$$

$$I_2 = \frac{1}{2} \sum_{B \in R}' m_B \left(\underset{a}{\underline{\omega}} \wedge \underset{b}{(\underline{x}_B - \underline{x}_0)}\right) \cdot \underset{c}{\left(\underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)\right)}$$

$$I_3 = \sum_{B \in R}' m_B \, \underline{v}_0 \cdot \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)$$

Guardiamo questi tre termini

- $I_1 = \frac{1}{2} \sum_{B \in R}' m_B \| \underline{v}_0 \|^2 = \frac{1}{2} M \| \underline{v}_0 \|^2$

  $\rightarrow$ dovuto alla traslazione

- I_2 : usiamo $\underline{a} \wedge \underline{b} \cdot \underline{c} = \underline{a} \cdot \underline{b} \wedge \underline{c}$

$$I_2 = \frac{1}{2} \underline{\omega} \cdot \sum_{B \in R} m_B (\underline{x}_B - \underline{x}_0) \wedge \left[\underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)\right]$$

$$= \frac{1}{2} \underline{\omega} \cdot I_0 (\underline{\omega})$$

trasformazione di inerzia

- $I_2 = \underline{v}_0 \cdot \underline{\omega} \wedge \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_0)$

$= \underline{v}_0 \cdot \underline{\omega} \wedge M (\underline{x}_G - \underline{x}_0)$

$(I_2 = 0$ per $O \equiv G)$

→ se O è un punto fisso : $\underline{v}_0 = 0$

→ se il rigido ruota intorno ad un asse fisso : $\underline{\omega} = \omega \underline{u}$

prendiamo $O \in$ asse

$K = \frac{1}{2} \underline{\omega} \cdot I_0(\underline{\omega}) = \frac{1}{2} \omega \underline{u} \cdot I_0(\omega \underline{u})$

$= \frac{1}{2} \omega^2 \underline{u} \cdot I_0 \cdot \underline{u} = \frac{1}{2} \omega^2 I_u$

→ se $\underline{\omega} = 0$ → $I_2, I_3 = 0$

Riassumendo

- moto generico

$K = \frac{1}{2} M \underline{v}_G^2 + \frac{1}{2} \underline{\omega} \cdot I_G(\underline{\omega})$

- moto rigido con punto fisso

$$K = \frac{1}{2}\, \underline{\omega} \cdot I_0(\underline{\omega})$$

- moto rigido con asse fisso (z)

$$K = \frac{1}{2} I_z \omega^2$$

- moto rigido di traslazione

$$K = \frac{1}{2} M v_G^2$$

<u>Sistemi rigidi piani</u> : in questo caso $\underline{\omega}$ è diretta lungo $\underline{e}_3$

$$\underline{\omega} \cdot I_0(\underline{\omega}) = \omega^2 \underline{e}_3 \cdot I_0(\underline{e}_3) =$$

$$= \omega^2 I_{33,0} = \omega^2 I_{3,0}$$

- moto rigido piano generico

$$K = \frac{1}{2} M v_G^2 + \frac{1}{2} I_{3,G} \omega^2$$

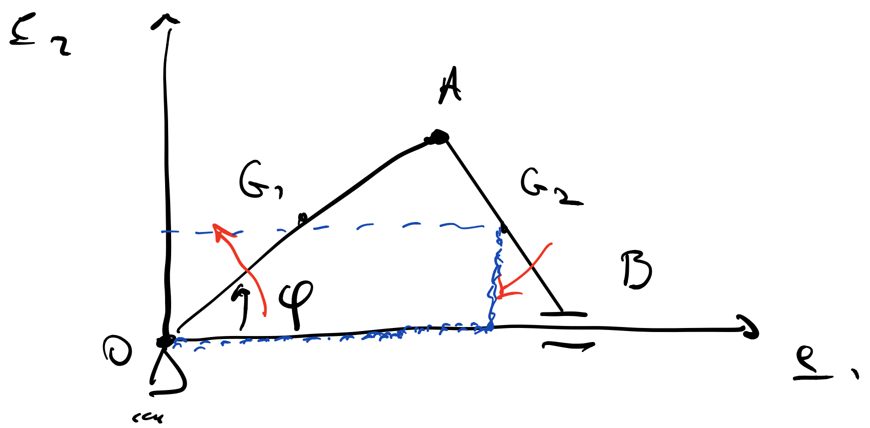
- moto rigido piano con punto O fisso

$$K = \frac{1}{2} I_{3,0} \omega^2$$

• moto rigido piano di traslazione

$$K = \frac{1}{2} M v_G^2$$

<u>Esempio</u>



aste omogenee di lunghezza l e massa m

$$K = K_{OA} + K_{AB}$$

• K_{OA} : OA ruota intorno ad O con velocità angolare $\dot\varphi e_3$

$$\rightarrow K_{OA} = \frac{1}{2} I_{3,O} \dot\varphi^2 = \frac{1}{2} \frac{ml^2}{3} \dot\varphi^2$$

• K_{AB} : AB moto rototraslatorio con velocità angolare

$$-\dot\varphi e_3$$

$$\rightarrow K_{AB} = \frac{1}{2} m v_{G_2}^2 + \frac{1}{2} \underbrace{\frac{ml^2}{12}}_{I_{G,3}} \dot\varphi^2$$

Calcoliamo v_G

$$\underline{x}_{G_2}(t) = \frac{3}{2}\, \ell \cos\varphi(\tau)\, \underline{e}_1 + \frac{\ell}{2} \sin\varphi(\tau)\, \underline{e}_2$$

$$\uparrow \qquad\qquad \uparrow$$

$$\varphi = \varphi(\tau) \qquad\qquad \varphi = \varphi(\tau)$$

$$\underline{v}_{G_2} = \frac{d}{d\tau}\underline{x}_{G_2} = \frac{\ell}{2}\Big(-3 \sin\varphi(\tau)\, \underline{e}_1 +$$

$$+ \cos\varphi(\tau)\, \underline{e}_2\Big) \frac{d}{d\tau}\varphi(\tau)$$

$$\|\underline{v}_{G_2}\|^2 = \frac{\ell^2}{4}\dot{\varphi}^2 \left(9 \sin^2\varphi + \cos^2\varphi\right)$$

$$= \frac{\ell^2}{4}\dot{\varphi}^2 \left(8 \sin^2\varphi + 1\right)$$

$$K = \frac{1}{2}\frac{m\ell^2}{3}\dot{\varphi}^2 + \frac{1}{2} m \frac{\ell^2}{4}\dot{\varphi}^2 \left(8 \sin^2\varphi + 1\right)$$

$$+ \frac{1}{2}\frac{m\ell^2}{12}\dot{\varphi}^2$$

$$= \frac{1}{2} m \ell^2 \dot{\varphi}^2 \left(\frac{2}{3} + 2 \sin^2\varphi\right)$$

$K, V \rightarrow \mathcal{L} = K - V$ e le eq. del moto.

Terza parte

Momento angolare

$$\underline{L}(0) = \sum_{B \in R}' (\underline{x}_B - \underline{x}_0) \wedge m_B \underline{v}_B$$

$$\Big(\underbrace{\underline{M}^e(0)}_{\uparrow} = \frac{d}{dt} \underline{L}(0) + \underline{v}_0 \wedge \underline{P} \Big) \qquad \underline{P} = M \underline{v}_G$$

1) Supponiamo che $O \in R$. Allora per Poisson $\underline{v}_B = \underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0)$

$$\underline{L}(0) = \sum_{B \in R}' (\underline{x}_B - \underline{x}_0) \wedge m_B \underline{v}_B$$

$$= \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_0) \wedge \left[\underline{v}_0 + \underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right]$$

$$= \left[\sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_0) \right] \wedge \underline{v}_0 +$$

$$+ \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_0) \wedge \left[\underline{\omega} \wedge (\underline{x}_B - \underline{x}_0) \right]$$

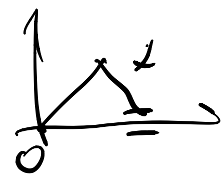
$$= M(\underline{x}_G - \underline{x}_O) \wedge \underline{v}_O + I_O(\underline{\omega})$$

In particolare

- $O \in R$ è fisso $\Rightarrow \underline{L}(O) = I_O(\underline{\omega})$
- $O \equiv G$ $\Rightarrow \underline{L}(G) = I_G(\underline{\omega})$

2) O non appartiene a R.

→ se abbiamo più rigidi



$O \notin R$

$A \in R$

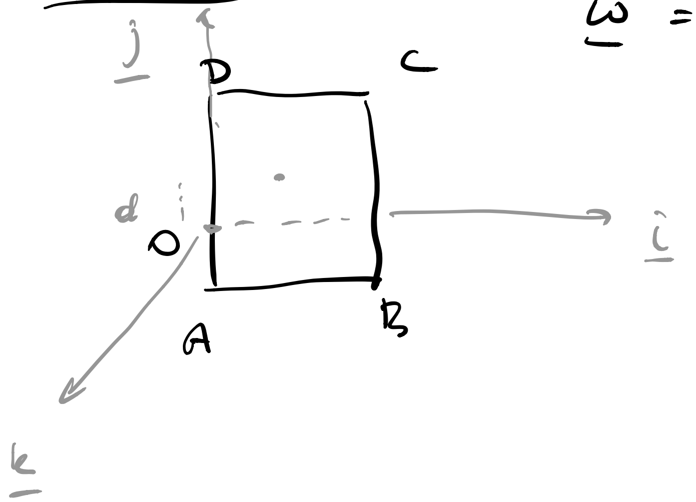
$$\underline{L}(O) = \sum_{B \in R}' (\underline{x}_B - \underline{x}_O) \wedge m_B \underline{v}_B$$

$$= \sum_{B \in R}' \left[(\underline{x}_B - \underline{x}_A) + (\underline{x}_A - \underline{x}_O) \right] \wedge m_B \underline{v}_B$$

$$= \underline{L}(A) + (\underline{x}_A - \underline{x}_O) \wedge \sum_{B \in R}' m_B \underline{v}_B$$

$$= \underline{L}(A) + (\underline{x}_A - \underline{x}_O) \wedge M \underline{v}_G$$

Esempio



$$\underline{\omega} = \dot{\varphi}\, \underline{j}$$

Prendiamo $S(O; \underline{i}, \underline{j}, \underline{k})$ origine in rotazione attorno a AD (cerniera cilindrica in O)

$$\underline{L}(O) = I_O(\underline{\omega}) =$$

$$= \begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\varphi} \\ 0 \end{pmatrix} = \begin{pmatrix} I_{12}\dot{\varphi} \\ I_{22}\dot{\varphi} \\ 0 \end{pmatrix}$$

$$= \boxed{I_{12}}\, \dot{\varphi}\, \underline{i} + I_{22}\, \dot{\varphi}\, \underline{j}$$

$\underline{j}$ non è asse principale di inerzia relativo ad O

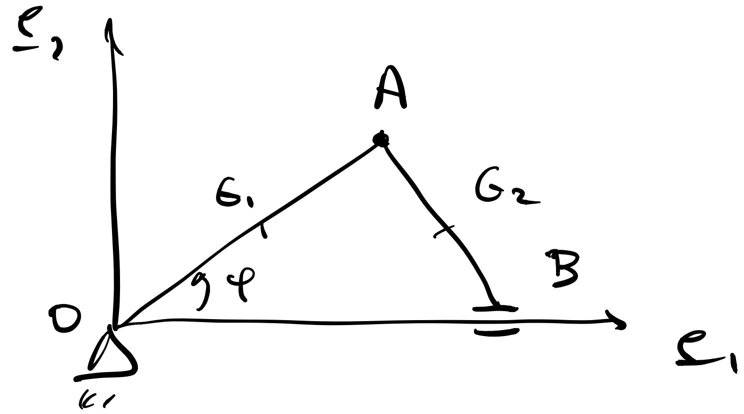
$\underline{L}(O)$ non è parallelo a $\underline{\omega}$

$$\left(\underline{M} = \frac{d}{dt} \underline{L} \right)$$

Nota : Rigidi piani : $\underline{\omega}$ è ortogonale al piano

$$\underline{L}(O) = I_O(\underline{\omega}) = I_{3,O}\,\underline{\omega}$$

Esempio



$$\underline{L}(O) = \underline{L}_{OA}(O) + \underline{L}_{AB}(O)$$

$$\underline{\omega} = \dot{\varphi}\,\underline{e}_3$$

$\underline{L}_{OA}(O)$: $O \in OA$

$$\Rightarrow \quad \underline{L}_{OA}(O) = I_O(\underline{\omega})$$

$$= \frac{m\ell^2}{3}\dot{\varphi}\,\underline{e}_3$$

$\underline{L}_{AB}(O)$: $O \notin AB$

$$\underline{L}_{AB}(O) = \underline{L}_{AB}(G_2) + (\underline{x}_{G_2} - \underline{x}_O) \wedge m\,\underline{v}_{G_2}$$

$$G_2 \in AB \rightarrow \underline{L}_{AB}(G_2) = I_{G_2}(\underline{\omega})$$

$$\underline{L}_{AB}(G_2) = -\frac{m\ell^2}{12}\dot{\varphi}\,\underline{e}_3$$

$$(\underline{x}_{G_2} - \underbrace{\underline{x}_0}_{=0}) \wedge m\,\underline{v}_{G_2} =$$

$$= \left(\frac{3}{2}\ell\cos\varphi\,\underline{e}_1 + \frac{\ell}{2}\sin\varphi\,\underline{e}_2\right) \wedge$$

$$\wedge\, m\left(-\frac{3}{2}\ell\sin\varphi\,\dot{\varphi}\,\underline{e}_1 + \frac{\ell}{2}\cos\varphi\,\dot{\varphi}\,\underline{e}_2\right)$$

$$= m\frac{3}{4}\ell^2\cos^2\varphi\,\dot{\varphi}\,\underline{e}_3 + m\frac{3}{4}\ell^2\sin^2\varphi\,\dot{\varphi}\,\underline{e}_3$$

$$= \frac{3}{4}\ell^2 m\,\dot{\varphi}\,\underline{e}_3$$

quindi

$$\underline{L}_{AB}(O) = -\frac{m\ell^2}{12}\dot{\varphi}\,\underline{e}_3 + \frac{3}{4}\ell^2 m\,\dot{\varphi}\,\underline{e}_3$$

$$\underline{L}(O) = \underline{L}_{OA}(O) + \underline{L}_{AB}(O)$$

$$= \frac{m}{3}\ell^2\dot{\varphi}\,\underline{e}_3 - \frac{m\ell^2}{12}\dot{\varphi}\,\underline{e}_3 + \frac{3}{4}\ell^2 m\,\dot{\varphi}\,\underline{e}_3$$

$$= m\ell^2\dot{\varphi}\left(\frac{1}{3} - \frac{1}{12} + \frac{3}{4}\right)\underline{e}_3$$

$$= m\ell^2\dot{\varphi}\,\underline{e}_3$$