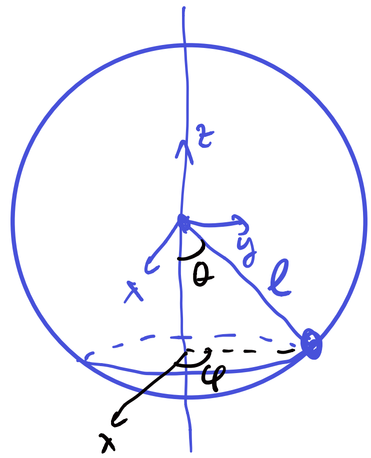
ESERCIZI

PENDOLO SFERICO



$\downarrow \bar{g}$

1) Prendere come coord. libere θ e φ scrivere L

$$\bar{r}(\bar{q}) : \begin{cases} x = \ell \sin\theta \cos\varphi \\ y = \ell \sin\theta \sin\varphi \\ z = -\ell \cos\theta \end{cases}$$

$$\bar{v}(\bar{q}, \dot{\bar{q}}) = \begin{cases} \dot{x} = \ell \dot{\theta} \cos\theta \cos\varphi - \ell \dot{\varphi} \sin\theta \sin\varphi \\ \dot{y} = \ell \dot{\theta} \cos\theta \sin\varphi + \ell \dot{\varphi} \sin\theta \cos\varphi \\ \dot{z} = \ell \dot{\theta} \sin\theta \end{cases}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m \left(\ell^2 \dot{\theta}^2 \underline{\cos^2\theta} + \ell^2 \dot{\varphi}^2 \sin^2\theta + \ell^2 \dot{\theta}^2 \underline{\sin^2\theta} \right)$$

$$= \frac{1}{2} m \left(\ell^2 \dot{\theta}^2 + \ell^2 \dot{\varphi}^2 \sin^2\theta \right)$$

$$V = mgz = -mg\ell \cos\theta$$

$$L = T - V = \frac{1}{2} m \ell^2 (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + mg\ell \cos\theta$$

MATRICE CINETICA:

$$T_2 = \frac{1}{2} \sum_{k,j} a_{kj} \dot{q}_k \dot{q}_j \qquad a = \begin{pmatrix} m\ell^2 & 0 \\ 0 & m\ell^2 \sin^2\theta \end{pmatrix}$$

$a = a(\theta)$

2) Ci sono coordinate cicliche? Se sì, qual è la cost. del moto?

φ è coord. ciclica $\left(\frac{\partial L}{\partial \varphi} = 0 \right)$

$$\frac{\partial L}{\partial \dot\varphi} = m\ell^2 \sin^2\theta\, \dot\varphi \quad \text{è una cost. del moto.}$$

3) Trovare la Lagrangiana ridotta (o efficace)

$$\tilde p_\varphi = m\ell^2 \sin^2\theta\, \dot\varphi \quad \to \quad \dot\varphi = \frac{\tilde p_\varphi}{m\ell^2 \sin^2\theta}$$

cost.

$$L^*(\theta, \dot\theta; \tilde p_\varphi) = L(\theta, \dot\theta, \dot\varphi)\Big|_{\dot\varphi = \frac{\tilde p_\varphi}{m\ell^2\sin^2\theta}} - \dot\varphi\, p_\varphi \Big|_{\dot\varphi = \frac{\tilde p_\varphi}{m\ell^2\sin^2\theta}} \quad (\Rightarrow p_\varphi = \tilde p_\varphi)$$

$$\left(L = T - V = \frac{1}{2} m\ell^2 (\dot\theta^2 + \dot\varphi^2 \sin^2\theta) + mg\ell\cos\theta \right)$$

$$L^* = \frac{1}{2} m\ell^2 \dot\theta^2 + \frac{1}{2} (m\ell^2 \cancel{\sin^2\theta}) \frac{\tilde p_\varphi^2}{(m\ell^2 \sin^2\theta)^{\cancel{2}}} + mg\ell\cos\theta$$

$$- \frac{\tilde p_\varphi}{m\ell^2 \sin^2\theta} \cdot \tilde p_\varphi$$

$$= \frac{1}{2} m\ell^2 \dot\theta^2 + mg\ell\cos\theta - \frac{\tilde p_\varphi^2}{2m\ell^2\sin^2\theta}$$

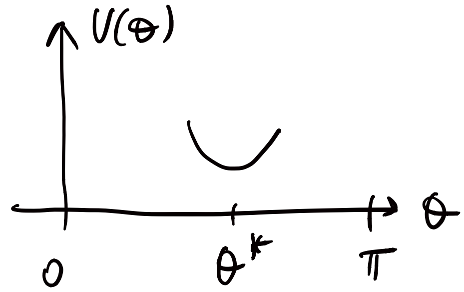
L^* è la Lagrangiana di un sistema 1 dimensionale con en. cinetica $T^* = \frac{1}{2} m\ell^2 \dot\theta^2$ e potenziale

$$V_{eff}(\theta) = -mg\ell\cos\theta + \frac{\tilde p_\varphi^2}{2m\ell^2\sin^2\theta}$$

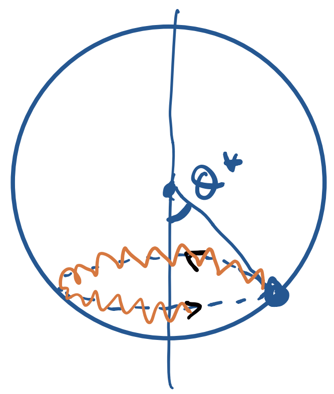
4) Pti di equil. del sist. efficace 1 dim.

$$V'_{eff}(\theta) = mgl\sin\theta + \frac{\tilde{p}_\varphi^2}{2ml^2}\left(\frac{-2\cos\theta}{\sin^3\theta}\right) = 0$$

$\exists\, \theta^*$ t.c. $V'_{eff}(\theta^*) = 0$



5) Cos'è pto fto di equil del sist. 1dim. nel sist. 2dim. originario?



$$\theta(t) = \theta^*$$

$$\dot\varphi(t) = \frac{\tilde{p}_\varphi}{ml^2\sin^2\theta(t)} = \frac{\tilde{p}_\varphi}{ml^2\sin^2\theta^*} \quad \text{cost.}$$

nel caso particolare

$$\Downarrow$$

$$\varphi(t) = \frac{\tilde{p}_\varphi}{ml^2\sin^2\theta^*}\cdot t + \varphi_0$$

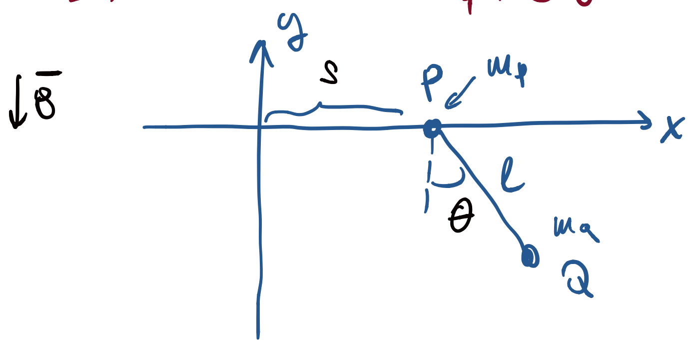
Le piccole oscillaz. di $\theta(t)$ attorno a θ^* mi danno un moto oscillatorio (vedi fig.)

p_φ coniugato a φ cioè è cost. del moto legata alla traj. $\varphi \to \varphi + c \Rightarrow p_\varphi$ è la componente del mom. ANG. $\vec{M}$ lungo l'asse zeta

$$p_\varphi = ml^2\sin^2\theta\,\dot\varphi$$

$$M_z = m x\dot{y} - m\dot{x}y = p_\varphi$$

ES. 2 del 24.06.19



$x_P = s \qquad \dot{x}_P = \dot{s}$

$y_P = 0 \qquad \dot{y}_P = 0$

$x_Q = s + \ell \operatorname{sen}\theta \qquad \dot{x}_Q = \dot{s} + \ell\dot{\theta}\cos\theta$

$y_Q = -\ell\cos\theta \qquad \dot{y}_Q = \ell\dot{\theta}\operatorname{sen}\theta$

1) $T = \frac{m_P}{2}(\dot{x}_P^2 + \dot{y}_P^2) + \frac{m_Q}{2}(\dot{x}_Q^2 + \dot{y}_Q^2) = \frac{m_P}{2}\dot{s}^2 + \frac{m_Q}{2}(\dot{s}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{s}\dot{\theta}\cos\theta)$

$= \frac{1}{2}M\dot{s}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\dot{s}\dot{\theta}\cos\theta$

$M = m_P + m_Q$
$m = m_Q$

$V = m_Q g y_Q = -mg\ell\cos\theta$

$L = \frac{1}{2}M\dot{s}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 + m\ell\dot{s}\dot{\theta}\cos\theta + mg\ell\cos\theta$

2) $T = \frac{1}{2}\left(M\dot{s}\dot{s} + m\ell^2\dot{\theta}\dot{\theta} + m\ell\cos\theta\,\dot{s}\dot{\theta} + m\ell\cos\theta\,\dot{\theta}\dot{s}\right)$

$\equiv \frac{1}{2}\left(a_{ss}\dot{s}\dot{s} + a_{\theta\theta}\dot{\theta}\dot{\theta} + a_{s\theta}\dot{s}\dot{\theta} + a_{\theta s}\dot{\theta}\dot{s}\right)$

$$a = \begin{pmatrix} M & m\ell\cos\theta \\ m\ell\cos\theta & m\ell^2 \end{pmatrix}$$

3) $\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = \frac{d}{dt}\left(M\dot{s} + m\ell\dot{\theta}\cos\theta\right) = M\ddot{s} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\operatorname{sen}\theta$

$\frac{\partial L}{\partial s} = 0$

$M\ddot{s} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\operatorname{sen}\theta = 0$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(m\ell^2\dot{\theta} + m\ell\dot{s}\cos\theta\right) = m\ell^2\ddot{\theta} + m\ell\ddot{s}\cos\theta - \cancel{m\ell\dot{s}\dot{\theta}\operatorname{sen}\theta}$

$\frac{\partial L}{\partial \theta} = -\cancel{m\ell\dot{s}\dot{\theta}\operatorname{sen}\theta} - mg\ell\operatorname{sen}\theta$

$\cancel{m}\ell^{\cancel{2}}\ddot{\theta} + \cancel{m}\cancel{\ell}\ddot{s}\cos\theta + \cancel{m}g\cancel{\ell}\operatorname{sen}\theta = 0$

$\ddot{\theta} = -\frac{\ddot{s}}{\ell}\cos\theta - \frac{g}{\ell}\operatorname{sen}\theta$

4) $L = \frac{1}{2} M \dot{s}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + m l \dot{s} \dot{\theta} \cos\theta + m g l \cos\theta$

L non dip. da s $\Rightarrow$ s è CICLICA

Cost. del moto è $\frac{\partial \mathcal{L}}{\partial \dot{s}} = M\dot{s} + m l \dot{\theta} \cos\theta$

5) $L^* = L - \dot{s}\tilde{p}_s \Big|_{\dot{s} = \frac{\tilde{p}_s - m l \dot{\theta}\cos\theta}{M}}$

$\tilde{p}_s = M\dot{s} + m l \dot{\theta} \cos\theta$

$\dot{s} = \frac{\tilde{p}_s}{M} - \frac{m}{M} l \dot{\theta} \cos\theta$

$$L^* = \frac{1}{2} M \frac{(\tilde{p}_s - m l \dot{\theta}\cos\theta)^2}{M^2} + \frac{1}{2} m l^2 \dot{\theta}^2 + m l \dot{\theta} \cos\theta \left(\frac{\tilde{p}_s - m l \dot{\theta}\cos\theta}{M}\right)$$
$$+ m g l \cos\theta - \tilde{p}_s \left(\frac{\tilde{p}_s - m l \dot{\theta} \cos\theta}{M}\right)$$

$$= \frac{1}{2} \frac{(\tilde{p}_s - m l \dot{\theta}\cos\theta)^2}{M} + \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos\theta$$
$$- \frac{(\tilde{p}_s - m l \dot{\theta}\cos\theta)^2}{M}$$

$$L^* = \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{(\tilde{p}_s - m l \dot{\theta}\cos\theta)^2}{2M} + m g l \cos\theta$$

$$= \frac{1}{2} m l^2 \dot{\theta}^2 \left(1 - \frac{m}{M}\cos^2\theta\right) - \overset{\text{cost}}{\cancel{\frac{\tilde{p}_s^2}{2M}}} + \underbrace{\boxed{\frac{\tilde{p}_s m l \dot{\theta} \cos\theta}{M}}}_{\text{valutata su un moto } \theta(t)} + m g l \cos\theta$$

"derivata totale"

$$\frac{\tilde{p}_s m l}{M} \dot{\theta}(t) \cos\theta(t) = \frac{d}{dt}\left(\frac{\tilde{p}_s m l \operatorname{sen}\theta(t)}{M}\right)$$

$\downarrow$ Possiamo utilizzare una Lagrangiana equiv.

$$L^* = \underbrace{\frac{1}{2} m l^2 \dot{\theta}^2 \left(1 - \frac{m}{M}\cos^2\theta\right)}_{T_{eff}} + \underbrace{mgl\cos\theta}_{-V_{eff}} \Rightarrow V_{eff} = -mgl\cos\theta$$

6) Pto equil. $V'_{eff}(\theta) = mgl\,\mathrm{sen}\,\theta = 0 \rightarrow \theta = 0, \pi$

$V''_{eff}(\theta) = mgl\cos\theta > 0$ per $\underline{\theta = 0}$ pto equil. STAB.

a) $\hat{L}^* = \frac{1}{2}\bar{\dot{q}} \cdot A\bar{\dot{q}} - \frac{1}{2}\bar{q} \cdot B\bar{q}$

$$A = a(\theta=0) = ml^2\left(1 - \frac{m}{M}\cos^2\theta\right)\Big|_{\theta=0} = ml^2\left(1 - \frac{m}{M}\right)$$

$$B = V''_{eff}(\theta=0) = mgl$$

$$\hat{L}^* = \frac{1}{2} ml^2\left(1 - \frac{m}{M}\right)\dot{\theta}^2 - \frac{1}{2} mgl\theta^2$$

b) $L^* = \frac{1}{2} m l^2 \dot{\theta}^2 \left(1 - \frac{m}{M}\cos^2\theta\right) + mgl\cos\theta$ espando in Taylor attorno a $(\theta, \dot{\theta}) = (0, 0)$

$$\hat{L}^* = \frac{1}{2} ml^2\dot{\theta}^2\left(1 - \frac{m}{M} + \dots\right) + mgl\left(1 - \frac{\theta^2}{2} + \dots\right)$$

$$= \frac{1}{2} ml^2\dot{\theta}^2\left(1 - \frac{m}{M}\right) - \frac{1}{2} mgl\theta^2$$

7) Freq. delle piccole osc. in sist. ridotto

a) $\det(B - \lambda A) = 0 \rightarrow mgl - \lambda\left(ml^2\left(1 - \frac{m}{M}\right)\right)$

$$\rightarrow \omega^2 = \lambda = \frac{mgl}{ml^2\left(1-\frac{m}{M}\right)} = \frac{g}{l}\,\frac{1}{1-\frac{m}{M}} > 0$$

b) Lag. linearizzata deve essere della forma

$$\frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}\mu\omega^2 x^2$$

$$\omega^2 = \frac{\text{coeff } x^2}{\text{coeff } \dot{x}^2} = \frac{\frac{1}{2}mgl}{\frac{1}{2}ml^2\left(1-\frac{m}{M}\right)} = \frac{g}{l}\,\frac{1}{1-\frac{m}{M}}$$