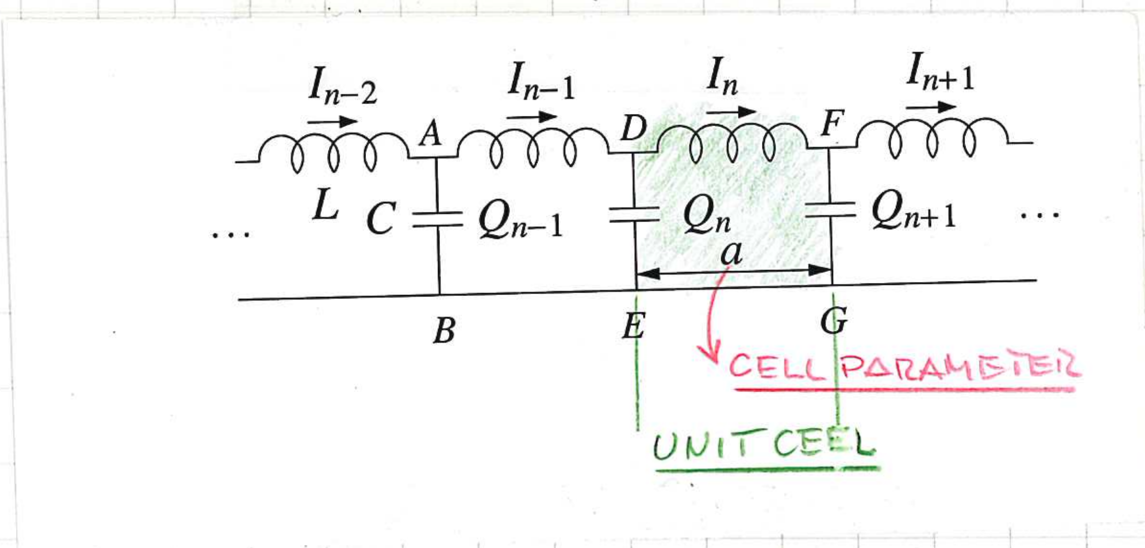
THE VALUE OF THE TRASLATIONAL SYMMETRY: FROM LC LADDER NETWORK TO PHONONS

THE IDEA THAT A PHYSICAL SYSTEM, OWING THE PROPERTY OF TRANSCLATIONAL SYMMETRY, CAN BE DESCRIBED BY THE PROPERTIES OF THE BASIC AND IRREDIUCIBLE ELEMENT, UNIT CELL, ON THE BASE OF WHICH THE ENTIRE SISTEM CAN BE BUILD UP, REGARDLESS ITS EXTENTION, IS A FUDAMENTAL CONCEPT AT THE BASE OF MANY STRUCTURAL THEORIES.

HERE MY INTENTION IS TO START BY DESCRIBING THE PROPERTIES OF A INDUCTANCE-CAPACITANCE (LC) INFINITE LADDER SYSTEM AN SHOW THAT IT CAN BE USE AS A TOY-MODEL TO EXPLAIN THE VIBRATIONAL MODES OF A CRYSTAL ALONG WITH ITS INTEACTION WITH LIGHT.

A LC LADDER NETWORK IS SHOWN IN THE FOLLOWING FIGURE



WE HAVE N INDUCTORS L AND N CAPACITOR C. N CAN BE EXTENDED TO ∞. WE DENOTE $I_n = I_n(t)$ THE CURRENT IN THE nTH INDUCTOR. RESISTENCE EFFECTS ARE ASSUMED TO BE NEGLIGIBLE. THE DISTANCE BETWEEN THE NEIGHBORING NODES IS "a". WE WANT TO FIND i) THE MECHANICAL EQUIVALENT OF THE SYSTEM, ii) TO SHOW I_n IS A SOLUTION OF THE MOTION EQUATION AND IT HAS THE FORM OF PROPAGATING MONOCHROMATIC WAVES $I_n = C e^{i(kna - \omega t)}$, (221)

iii) TO DEMONSTRATE THAT EXIST A DISPERTION RELATION BETWEEN K AND ω; iV) DISCUSS THE LIMIT TO A CONTINUUM SYSTEM $N \to \infty$, $n \to \infty$, $a \to 0$ WITH $na \to x$.

LET Q_n BE THE CHARGE ON THE nTH CAPACITOR, KIRCHHOFF'S JUNCTION RULE AT THE JUNCTION D OF THE FIGURE IMPLIES

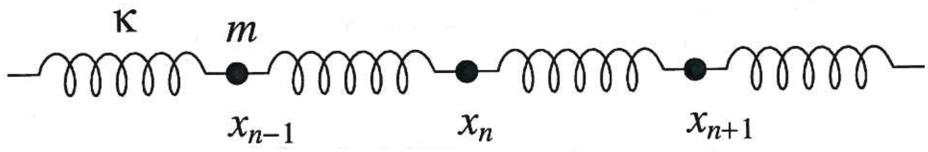
$$\frac{dQ_n}{dt} = I_{n-1} - I_n \quad (222)$$

WHILE KIRCHHOFF MESH RULE APPLIED TO MESH DEFG IMPLIES $\frac{Q_n}{C} - \frac{Q_{n+1}}{C} = L\frac{dI_n}{dt}$ (223)

BY DIFFERENTIATING 171 WITH RESPECT TO TIME AND INSERT 170 FOR THE DERIVATIVES OF Q_n WE OBTAIN $\frac{d^2 I_n}{dt^2} = \omega_0^2 (I_{n-1} - 2I_n + I_{n+1})$ (224) WHERE

$$\omega_0^2 = \frac{1}{LC}$$

THE EQUIVALENT MECHANICAL SYSTEM IS A LINEAR SEQUENCE OF N IDENTICAL MASSES m, BEING THE MASSES CONNECTED THROUGH A SPRING OF CONSTANT k



WE DENOTE BY X_n THE DISPLACEMENT OF EACH MASS FROM EQUILIBRIUM (SPRINGS AT REST), THUS THE EQS. OF MOTIONS OF THE NTH MASS IS

$$m\frac{d^2x_n}{dt^2} = -k(x_n - x_{n-1}) + k(x_{n+1} - x_n) \quad (173)$$

DIVIDING BY m AND SETTING $\omega_0^2 = \frac{k}{m}$ WE OBTAIN

$$\frac{d^2x_n}{dt^2} = \omega_0^2(x_{n-1} - 2x_n + x_{n+1}) \quad (225)$$

WHICH IS MATHEMATICALLY EQUIVALENT TO 172. THIS EQ. CAN BE GENERALIZED TO THE CASE OF MECHANICAL SYSTEMS WHERE TRASVERSE DISPLACEMENTS ARE ALLOWED ALSO. IN 3D $\bar{r}_n \Rightarrow$

$$\frac{d^2\bar{r}}{dt^2} = \omega_0^2(\bar{r}_{n-1} - 2\bar{r}_n + \bar{r}_{n+1}). \quad (226)$$

IF WE ASSUME A $\bar{k} > 0$ (PROGRESSIVE WAVE). BY USING 169 IN 172 AND DIVIDING BOTH SIDES

BY $Ce^{-i\omega t}$ WE OBTAIN

$$-\omega^2 e^{ikna} = \omega_0^2 \left[e^{ik(n+1)a} - 2e^{ikna} + e^{ik(n-1)a} \right]$$ 227

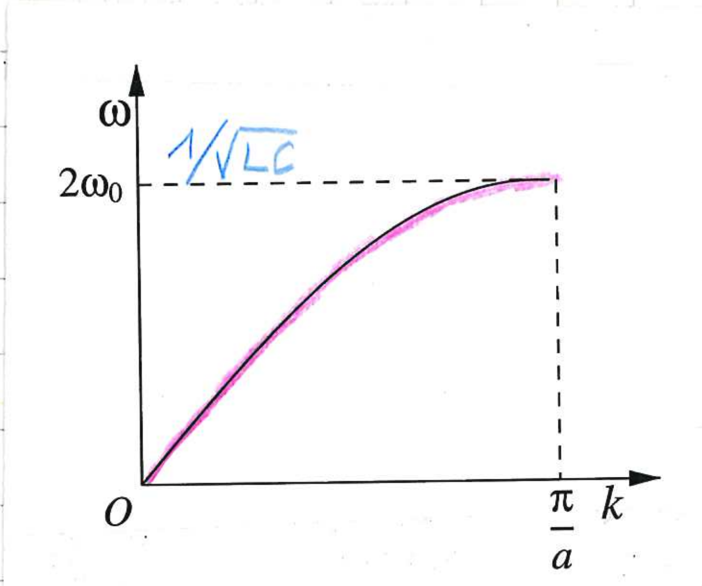
DIVIDING BOTH SIDES BY e^{ikna} WE OBTAIN

$$\omega^2 = \omega_0^2 \left(2 - e^{ika} - e^{-ika}\right) =$$

$$= 2\omega_0^2 (1 - \cos ka) = 4\omega_0^2 \sin^2(ka/2)$$ or

PERFORMING THE ROOT SQUARE

$$\omega = 2\omega_0 \left| \sin\left(\frac{ka}{2}\right) \right| \quad (228)$$



THIS FIGURE SHOWS THE 228 DISPERSION RELATION FOR $0 < k < \pi/a$. IT IS EASY TO UNDERSTAND THAT WHY THE LC LADDER CIRCUIT IS REPPRESENTED IN THE DIRECT SPACE (LATTICE PARAMETER "a") THE DISPERSION RELATION IS IN THE RECIPROCAL SPACE $1/a$.

- OBSERVATION FROM THIS RELATION WE CAN DRAW ONE OF THE MOST IMPORTANT FINDING

FOR DECRIBING THE PHYSICAL BEHAVIOUR OF A TRANSLATIONAL SYMMETRY SYSTEM. IN THE RECIPROCAL SPACE THE K RANGE $0<K<\pi/a$ IS SUFFICIENT TO DESCRIBE ALL THE WAVE PROPAGATING IN THE SYSTEM, REGARDLESS THE NUMBER N OF THE UNIT CELLS. IN FACT, ALTHOUGH 228 SEEMS TO IMPLY THAT $\omega(k)$ IS A PERIODIC FUNCTION OF K WITH PERIOD $2\pi/a$ THE WAVE VECTOR $\bar{K}$ AND $\bar{K}'=\bar{K}+2\pi s/a$, BEING S AN INTEGER, ACTUALLY THEY REPRESENT THE SAME WAVE, SINCE

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$$e^{ik'a}=e^{i(k+2\pi s/a)na}=e^{ikna}e^{i2\pi sn}=e^{ikna}$$

"sn" BEING INTEGER. THIS IS WHY IT IS SUFFICIENT TO CONSIDER THE RANGE $0<K<2\pi/a$.

- OBSERVATION THE EXSTENCE OF A MAX WAVE VECTOR AND OF A CUTOFF FREQUENCY IS RELATED TO THE DISCRETE NATURE OF THE NETWORK WHICH IMPOSES A MINIMUM SAMPLING RATE a, THE MAX K VALUE $K_{MAX}=\pi/a$ CORRESPOND TO $\lambda_{min}=2\pi/K_{MAX}=2a$. A WAVE WITH A SMALLER λ IS NOT SUPPORTED BY THE SYSTEM. FOR THE SUPPORTED WAVE THE CURRENT $(I(t))$ INTENSITY VALUE IS REPEATED EVERY TWO MESHES OF THE NETWORK AS SHOW IN THE FOLLOWING FIGURE. FINALLY THE K_{MAX} $\Rightarrow$ A CUTOFF

FREQUENCY $\omega_{MAX} = 2\omega_0 \Rightarrow \omega > \omega_{MAX}$ CANNOT BE TRASMITTED A LC NETWORK ACTS AS A LOW-PASS FILTER