9.1.5 Local character and support of a distribution

The following results says that it is sufficient to know the behavior of a distribution in a neighborhood of each point, to know its behavior in general.

Theorem 35. *Let T_1 and T_2 be two distributions in $\mathcal{D}'(\Omega)$. Suppose that, for all $x_0 \in \Omega$, there exists a neighborhood U_0 of x_0 such that, for all $\varphi \in \mathcal{D}(\Omega)$, if $\operatorname{Supp}\varphi \subseteq U_0$, then $T_1(\varphi) = T_2(\varphi)$.*

Then $T_1 = T_2$.

Proof. Let $\psi \in \mathcal{D}(\Omega)$. Denote by K the support of ψ. We know that, for all $x \in K$, there exists an open neighborhood U_x of x such that, for all $\varphi \in \mathcal{D}(\Omega)$, if $\operatorname{Supp}\varphi \subseteq U_x$, then $T_1(\varphi) = T_2(\varphi)$. Form the open covering $\{U_x \mid x \in K\}$ we extract a finite subcovering of K,

$$U_1,\ U_2, \ldots, U_N.$$

We use now the theorem on partition of unity (Theorem 32 in Lesson 9). There exist $\varphi_1, \ldots, \varphi_N$ in $\mathcal{D}(\Omega)$, with, for all j, $\operatorname{Supp}\varphi_j \subseteq U_j$, such that, for all $x \in K$, $\sum_j \varphi_j(x) = 1$. Then

$$\begin{aligned} T_1(\psi) &= T_1(\psi \textstyle\sum_j \varphi_j) \\ &= T_1(\textstyle\sum_j \psi\varphi_j) \\ &= \textstyle\sum_j T_1(\psi\varphi_j) \qquad \text{with} \quad \operatorname{Supp}\psi\varphi_j \subseteq U_j \\ &= \textstyle\sum_j T_2(\psi\varphi_j) \\ &= T_2(\textstyle\sum_j \psi\varphi_j) \\ &= T_2(\psi \textstyle\sum_j \varphi_j) = T_2(\psi). \end{aligned}$$

□

Now we define the support of a distribution.

Definition 27. *Let $T \in \mathcal{D}'(\Omega)$. Let $x \in \Omega$. We say that $x \notin \operatorname{Supp} T$ if there exists a neighborhood U of x such that, for all $\varphi \in \mathcal{D}(\Omega)$, if $\operatorname{Supp}\varphi \subseteq U$, then $T(\varphi) = 0$. $\operatorname{Supp} T$ is the smallest relatively closed set in Ω outside of which T is identically equal to 0.*

Remark 20. *Let $f \in C(\Omega)$. Then $f \in L^1_{loc}(\Omega)$ and consequently we can consider the distribution T_f associated to f. The support of f as continuous function coincides with the support of f as L^1_{loc} function and with the support of T_f as distribution.*

10 Lesson 11 – April 12^{th}, 2021

10.1 Derivative of a distribution, multiplication of a distribution with smooth function

The content of this paragraph can be found in [6, Ch. 1.4] (see also [12]).

Derivative in the sense of distributions

def Let $T \in \mathcal{D}'(\Omega)$

we define $\partial_{x_j} T$ as an operator acting on $\mathcal{D}(\Omega)$

$$\partial_{x_j} T(\varphi) = -T(\partial_{x_j}\varphi)$$

$\uparrow$ we are defining this

$\uparrow$ we know this (is the classical derivative)

Then $\partial_{x_j} T$ is a distribution (we call it the derivative (in sense of distributions) of the distribution T)

- $\partial_{x_j} T$ is linear ($\partial_{x_j} T(a\varphi + b\psi) = a \cdot \partial_{x_j} T(\varphi) + b\,\partial_{x_j} T(\psi)$)

exercise.

usual

- $\partial_{x_j} T$ should verify the inequality of distributions.

we know that T is distribution

so fix K compact set in Ω, $\exists C_K, m_K$

s.t. $\forall \varphi \in \mathcal{D}(\Omega)$ with supp in K, $|T(\varphi)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_\Omega |\partial^\alpha \varphi|$

so that $|\partial_{x_j} T(\varphi)| = |T(\partial_{x_j}\varphi)| \leq C_K \sum_{|\alpha| \leq m_K} \sup_\Omega \partial^\alpha(\partial_{x_j}\varphi)$

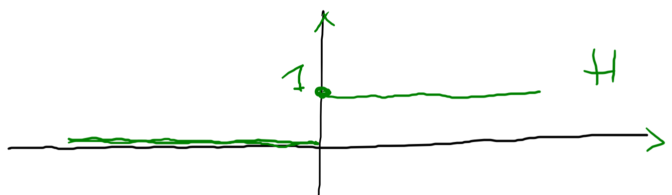
$$\leq C_K \sum_{|\beta| \leq m_K + 1} \sup_\Omega |\partial^\beta \varphi|$$

$\partial_{x_j} T$ is a distribution with possibly order $m+1$

(if T had order m)

Ex. Heaviside function

$H : \mathbb{R} \to \mathbb{R}$ $\qquad H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$



H is an L^1_{loc} function

We associate to H a distribution T_H

¿ what about H' ?

$H'(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \text{does not exist} & \text{if } x = 0 \end{cases}$

↑ classical derivative

$2H$ → the same

¿ what about T_H' ?

↑ (in sense of distr.)

$\varphi \in C_0^\infty(\mathbb{R})$

$$(T_H)'(\varphi) = T_H(-\varphi') = -\int_{-\infty}^{+\infty} H(x) \cdot \varphi'(x)\,dx$$

$\varphi \in C_0^\infty(]-M, M[)$

$$= -\int_0^{+\infty} \varphi'(x)\,dx = -\varphi(x)\Big|_0^{+\infty} = \varphi(0)$$

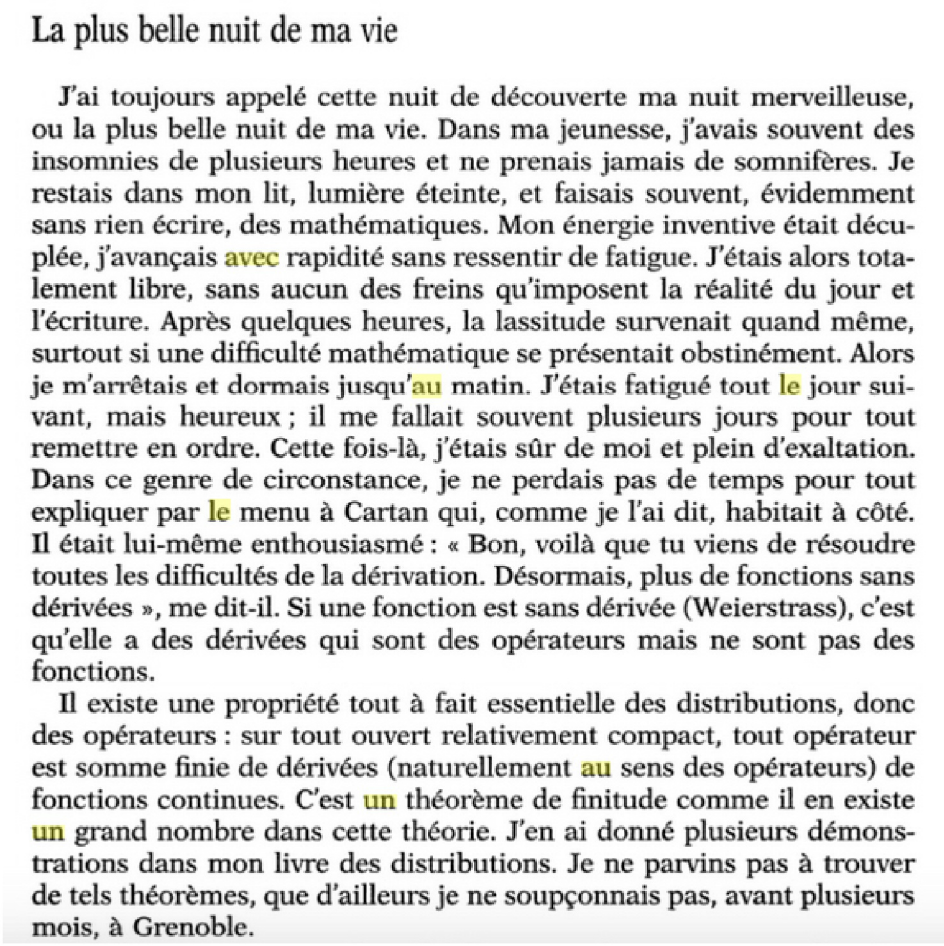
(not important → $+\infty$)

$$= \delta_0(\varphi)$$

$$\boxed{H'^{(d)} = \delta_0}$$

rem: δ_0 distribution with support in $\{0\}$; it is identically $= 0$ in $x \neq 0$

$(2H)'^{(d)} = 2\delta_0$



La plus belle nuit de ma vie

J'ai toujours appelé cette nuit de découverte ma nuit merveilleuse, ou la plus belle nuit de ma vie. Dans ma jeunesse, j'avais souvent des insomnies de plusieurs heures et ne prenais jamais de somnifères. Je restais dans mon lit, lumière éteinte, et faisais souvent, évidemment sans rien écrire, des mathématiques. Mon énergie inventive était décuplée, j'avançais avec rapidité sans ressentir de fatigue. J'étais alors totalement libre, sans aucun des freins qu'imposent la réalité du jour et l'écriture. Après quelques heures, la lassitude survenait quand même, surtout si une difficulté mathématique se présentait obstinément. Alors je m'arrêtais et dormais jusqu'au matin. J'étais fatigué tout le jour suivant, mais heureux ; il me fallait souvent plusieurs jours pour tout remettre en ordre. Cette fois-là, j'étais sûr de moi et plein d'exaltation. Dans ce genre de circonstance, je ne perdais pas de temps pour tout expliquer par le menu à Cartan qui, comme je l'ai dit, habitait à côté. Il était lui-même enthousiasmé : « Bon, voilà que tu viens de résoudre toutes les difficultés de la dérivation. Désormais, plus de fonctions sans dérivées », me dit-il. Si une fonction est sans dérivée (Weierstrass), c'est qu'elle a des dérivées qui sont des opérateurs mais ne sont pas des fonctions.

Il existe une propriété tout à fait essentielle des distributions, donc des opérateurs : sur tout ouvert relativement compact, tout opérateur est somme finie de dérivées (naturellement au sens des opérateurs) de fonctions continues. C'est un théorème de finitude comme il en existe un grand nombre dans cette théorie. J'en ai donné plusieurs démonstrations dans mon livre des distributions. Je ne parvins pas à trouver de tels théorèmes, que d'ailleurs je ne soupçonnais pas, avant plusieurs mois, à Grenoble.

Figure 16: Page 243 of Laurent Schwartz's autobiography [13]

10.1.1 Derivative of a distribution

Let $f \in C^1(\Omega)$. We notice that both f and $\partial_{x_j} f$ are in $L^1_{loc}(\Omega)$, so we can consider the associated distributions, i. e., for $\varphi \in \mathcal{D}(\Omega)$,

$$T_f(\varphi) = \int_\Omega f\varphi \qquad \text{and} \qquad T_{\partial_{x_j} f}(\varphi) = \int_\Omega (\partial_{x_j} f)\varphi.$$

But

$$T_{\partial_{x_j} f}(\varphi) = \underbrace{\int_\Omega (\partial_{x_j} f)\varphi = -\int_\Omega f(\partial_{x_j}\varphi)}_{\text{integration by parts}} = -T_f(\partial_{x_j}\varphi).$$

Consequently, if you want that a (to be defined) derivative, with respect to x_j, of the distribution T_f, associated to f, behaves like the distribution $T_{\partial_{x_j} f}$, associated to the classical derivative of f, you have to set

$$(\partial_{x_j} T_f)(\varphi) = -T_f(\partial_{x_j}\varphi).$$

Definition 28. *Let $T \in \mathcal{D}'(\Omega)$. For all $\varphi \in \mathcal{D}(\Omega)$, we define*

$$(\partial_{x_j} T)(\varphi) = -T(\partial_{x_j}\varphi).$$

Theorem (Structure of a distribution)

Let Ω be an open set. Let ω be a relatively compact open subset of Ω (i.e. $\bar{\omega}$ is compact set contained in Ω)

Let $T \in \mathcal{D}'(\Omega)$.

then $\exists f \in L^\infty(\omega)$ s.t. for all $\varphi \in \mathcal{D}(\omega)$

$\exists m \in \mathbb{N}$, $T(\varphi) = \int_\omega f \cdot D_1^m D_2^m \cdots D_n^m \varphi \, dx$

i.e. T on $\mathcal{D}(\omega)$ is $T = (-1)^{nm} D_1 \cdots D_n (T_f)$

locally T is the derivative (in the sense of distribution) of a distribution associated to a L^∞ function.

proof. Suppose $T(\varphi) = \int_\omega f \, D_1^m \cdots D_n^m \varphi \, dx$ $\quad \left(D_j = (-1) \frac{\partial}{\partial x_j} \right)$

where $f \in L^\infty(\omega)$ $\quad \forall \varphi \in \mathcal{C}_c^\infty(\omega)$

we have that

$$|T(\varphi)| \leq \|f\|_{L^\infty(\omega)} \int_\omega |D_1^m \cdots D_n^m \varphi| \, dx$$

$$\forall \varphi \in \mathcal{D}(\omega), \quad |T(\varphi)| \leq C \cdot \|D_1^m \cdots D_n^m \varphi\|_{L^1(\omega)} \quad (*)$$

I show the converse.

if $(*)$ is valid then the thesis of theorem is true

Suppose that $(*)$ is true.

consider $V = \{ D_1^m D_2^m \cdots D_n^m \varphi : \varphi \in \mathcal{C}_0^\infty(\omega) \}$

(in principle this is in $\mathcal{C}_0^\infty(\omega)$)

obviously $V \subseteq L^1(\omega)$ $\quad$ V is subspace of $L^1(\omega)$

Consider $\Phi : V \longrightarrow \mathbb{R}$ (or $\mathbb{C}$)

linear $\quad$ ($V \subseteq L^1(\omega)$)

$$D_1^m \cdots D_n^m \varphi \longmapsto \Phi(D_1^m \cdots D_n^m \varphi) = T(\varphi)$$

I remark that Φ is continuous w.r.t. norm of $L^1(\omega)$

because $|T(\varphi)| \leq C \|D_1^m \cdots D_n^m \varphi\|_{L^1(\omega)}$ by $(*)$

Apply Hahn-Banach.

Φ can be extended to $L^1(\omega)$

$\exists \tilde{\Phi} : L^1(\omega) \to \mathbb{R}$ (or $\mathbb{C}$) $\quad \tilde{\Phi} \in (L^1(\omega))' = L^\infty(\omega)$ (Riesz)

and $\tilde{\Phi} = \Phi$ on V.

$\exists g \in L^\infty(\omega)$ s.t. $\tilde{\Phi}(h) = \int_\omega g h$

so $\Phi(D_1^m \cdots D_n^m \varphi) = \int_\omega g \, D_1^m \cdots D_n^m \varphi$

$\| T(\varphi)$

$$T(\varphi) = \int_\omega g \, D_1^m \cdots D_n^m \varphi$$

To conclude the proof it is sufficient to prove that $\omega \subset\subset \Omega$

$\varphi,\ T \in \mathcal{D}'(\Omega)$

then $\exists\, m,\ \exists\, C$ s.t. (*)

$$|T(\varphi)| \le C \, \| D_1^{m} \dots D_n^{m} \varphi \|_{L^1(\omega)} \quad \forall \varphi \in \mathcal{D}(\omega)$$

We know that T is a distribution and $\bar{\omega}$ is compact set

We apply the definition of dist. with $K = \bar{\omega}$

$\exists\, C_{\bar{\omega}},\ m_{\bar{\omega}}$ s.t.

$\forall \varphi \in C_0^\infty(\Omega)$ with support in $\bar{\omega}$ we have

$$|T(\varphi)| \le C_{\bar{\omega}} \sum_{|\alpha| \le m_{\bar{\omega}}} \sup_{\Omega} |\partial^\alpha \varphi| \quad (**)$$

remark: $\varphi \in C_0^\infty(\mathbb{R})$ supp $\varphi \subseteq [-M, M]$

$$\varphi(x) = \int_{-\infty}^{x} \varphi'(t)\,dt = \int_{-M}^{x} \varphi'(t)\,dt$$

$$\forall x \quad |\varphi(x)| \le \int_{-M}^{M} |\varphi'(t)|\,dt \le 2M \cdot \sup |\varphi'|$$

$$\sup_{x \in \Omega} |\varphi(x)| \le 2M \sup_{x \in \Omega} |\varphi'(x)|$$

depends on diameter of $\bar{\omega}$ and on $m_{\bar{\omega}}$

$\forall \alpha$ with $|\alpha| \le m_{\bar{\omega}}$

$$\sup_{x \in \Omega} |D^\alpha \varphi| \le C \sup_{x \in \Omega} |D_1^{m_\omega} \dots D_n^{m_\omega} \varphi|$$

$$(**) \Rightarrow \quad |T(\varphi)| \le C \cdot \underbrace{\sup_{x \in \Omega} |D_1^{m_\omega} \dots D_n^{m_\omega} \varphi|}_{\| D_1^{m_{\bar{\omega}}} \dots D_n^{m_{\bar{\omega}}} \varphi \|_{L^\infty(\omega)}}$$

to finish it sufficient to consider that

$$|D_1^{m_{\bar{\omega}}} \dots D_n^{m_{\bar{\omega}}} \varphi(x)| \le \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \dots \int_{-\infty}^{x_n} |D_1^{m_{\bar{\omega}}+1} \dots D_n^{m_{\bar{\omega}}+1} \varphi(y)|\,dy$$

$$\Downarrow$$

$$\| D_1^{m_{\bar{\omega}}} \dots D_n^{m_{\bar{\omega}}} \varphi \|_{L^\infty} \le (\operatorname{diam} \bar{\omega})^n \, \| D_1^{m_{\bar{\omega}}+1} \dots D_n^{m_{\bar{\omega}}+1} \varphi \|_{L^1(\omega)}$$

QED

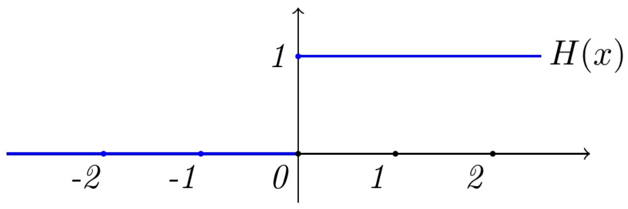
We have that $\partial_{x_j} T \in \mathcal{D}'(\Omega)$, in fact $\partial_{x_j} T$ is linear and

$$|(\partial_{x_j} T)(\varphi)| = |T(\partial_{x_j}\varphi)| \le C_K \sum_{|\alpha|\le m_K} \sup_\Omega |D^\alpha(\partial_{x_j}\varphi)| \le C_K \sum_{|\beta|\le m_K+1} \sup_\Omega |D^\beta \varphi|$$

for all $\varphi \in \mathcal{D}(\Omega)$ such that $\operatorname{Supp}\varphi \subseteq K$. Remark that, if T is a distribution of order m, then $\partial_{x_j} T$ is a distribution of order less or equal to $m+1$.

Example 8. *Let H be the Heavisde function, i. e.*

$$H : \mathbb{R} \to \mathbb{R}, \qquad H(x) = \begin{cases} 1 & \text{for} \quad x \ge 0, \\ 0 & \text{for} \quad x < 0. \end{cases}$$



H is a L^1_{loc} function. We denote by H also the associated distribution, i. e.

$$H : \mathcal{D}(\mathbb{R}) \to \mathbb{R}, \qquad H(\varphi) = \int_{\mathbb{R}} H\varphi = \int_0^{+\infty} \varphi(t)\, dt.$$

Let's compute the derivative of H as a distribution.

$$H'(\varphi) = -H(\varphi') = -\int_0^{+\infty} \varphi'(t)\, dt = -\varphi(t)\big|_0^{+\infty} = \varphi(0) = \delta_0(\varphi),$$

i. e. $H' = \delta_0$, the derivative of the Heaviside distribution is Dirac's delta at 0. Remark that the Heaviside function possess finite classical derivative equal to 0 for all $x \in \mathbb{R} \setminus \{0\}$. The derivative in the sense of distribution is more precise: Dirac's delta at 0 coincide, as distribution, to 0 in a neighborhood of each point of $\mathbb{R} \setminus \{0\}$, but gives a precise information also at 0.

Exercise 4. *Consider, for all $\varphi \in \mathcal{D}(\mathbb{R})$,*

$$PV_{\frac{1}{x}}(\varphi) = \lim_{\varepsilon \to 0^+} \int_{|x|\ge\varepsilon} \frac{\varphi(x)}{x}\, dx.$$

Prove that $PV_{\frac{1}{x}}$ is a distribution of order ≤ 1 (we call it principal value of $\frac{1}{x}$).

Denote by $T_{\log}$ the distribution associated to the $L^1_{loc}(\mathbb{R})$ function $x \mapsto \log|x|$. Prove that $T'_{\log} = PV_{\frac{1}{x}}$.

Hint. Let $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{Supp}\varphi \subseteq [-M, M]$. Remark that in this case

$$PV_{\frac{1}{x}}(\varphi) = \lim_{\varepsilon \to 0^+} \int_{\varepsilon\le|x|\le M} \frac{\varphi(x)}{x}\, dx.$$

Remarking that

$$\int_{\varepsilon\le|x|\le M} \frac{\varphi(0)}{x}\, dx = \varphi(0) \int_{\varepsilon\le|x|\le M} \frac{1}{x}\, dx = 0,$$

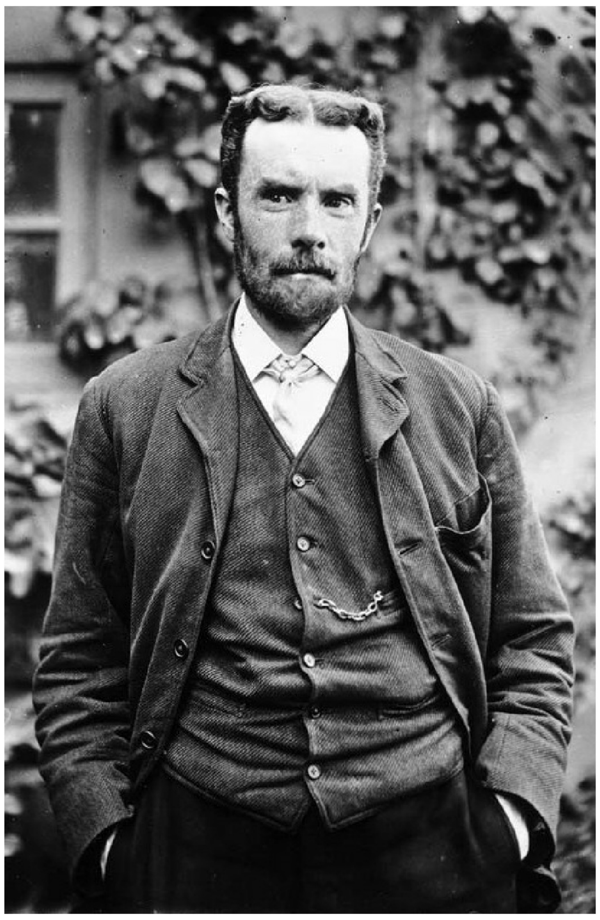


Figure 17: Olivier Heaviside (1850-1925)

we have that

$$\int_{\varepsilon \le |x| \le M} \frac{\varphi(x)}{x}\, dx = \int_{\varepsilon \le |x| \le M} \frac{\varphi(x) - \varphi(0)}{x}\, dx.$$

Consider now the function

$$\psi(x) = \begin{cases} \dfrac{\varphi(x) - \varphi(0)}{x} & \text{if } x \neq 0, \\ \varphi'(0) & \text{if } x = 0. \end{cases}$$

We have that $\psi \in C([-M,\, M])$. Consequently

$$\lim_{\varepsilon \to 0^+} \int_{|x| \ge \varepsilon} \frac{\varphi(x)}{x}\, dx = \lim_{\varepsilon \to 0^+} \int_{\varepsilon \le |x| \le M} \psi(x)\, dx = \int_{-M}^{M} \psi(x)\, dx,$$

so that the limit exists and it is finite. Moreover

$$\left| \int_{-M}^{M} \psi(x)\, dx \right| \le 2M \sup_{[-M,M]} |\psi| \qquad \text{and} \qquad \sup_{[-M,M]} |\psi| \le \sup_{\mathbb{R}} |\varphi'|.$$

We obtain finally

$$|PV_{\frac{1}{x}}(\varphi)| \le 2M \sup_{\mathbb{R}} |\varphi'| \quad \text{for all } \ \varphi \in \mathcal{D}(\mathbb{R}) \ \text{ with } \ \operatorname{Supp} \varphi \subseteq [-M,\, M],$$

i. e. $PV_{\frac{1}{x}}$ is a distribution of order ≤ 1.

Consider, for $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{Supp}\varphi \subseteq [-M, M]$,

$$T_{\log}(\varphi) = \int_{-M}^{M} (\log|x|)\varphi(x)\,dx.$$

We have

$$T'_{\log}(\varphi) = -T_{\log}(\varphi') = -\int_{-M}^{M} (\log|x|)\varphi'(x)\,dx.$$

Since the function $x \mapsto (\log|x|)\varphi'(x)$ is a L^1 function, we have that

$$\int_{-M}^{M} (\log|x|)\varphi'(x)\,dx = \lim_{\varepsilon\to 0^+}(\int_{-M}^{-\varepsilon} (\log|x|)\varphi'(x)\,dx + \int_{\varepsilon}^{M} (\log|x|)\varphi'(x)\,dx).$$

Now

$$\int_{-M}^{-\varepsilon} (\log|x|)\varphi'(x)\,dx = (\log|\varepsilon|)\varphi(-\varepsilon) - \int_{-M}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx$$

and

$$\int_{\varepsilon}^{M} (\log|x|)\varphi'(x)\,dx = -(\log|\varepsilon|)\varphi(\varepsilon) - \int_{\varepsilon}^{M} \frac{\varphi(x)}{x}\,dx.$$

We finally obtain

$$\int_{-M}^{M} (\log|x|)\varphi'(x)\,dx = \lim_{\varepsilon\to 0^+}[(\log|\varepsilon|)(\varphi(-\varepsilon) - \varphi(\varepsilon)) - \int_{|x|\le\varepsilon} \frac{\varphi(x)}{x}\,dx]$$

and the conclusion follows. Remark that the second part of the exercise already contains the first part, i. e. if one proves that $PV_{\frac{1}{x}}$ is the derivative of a distribution of order 0, then $PV_{\frac{1}{x}}$ is immediately a distribution of order ≤ 1.

Remark 21. *The function $x \mapsto \frac{1}{x}$ is not a $L^1_{loc}(\mathbb{R})$ function, so that it is not possible to define a distribution associated to this function. The distribution $PV_{\frac{1}{x}}$ is the correct substitute.*

Exercise 5. *Consider, for all $\varphi \in \mathcal{D}(\mathbb{R})$,*

$$FP_{\frac{1}{x^2}}(\varphi) = \lim_{\varepsilon\to 0^+}(\int_{|x|\ge\varepsilon} \frac{\varphi(x)}{x^2}\,dx - 2\frac{\varphi(0)}{\varepsilon}).$$

Prove that $FP_{\frac{1}{x^2}}$ is a distribution of order ≤ 2 (we call it finite part of $\frac{1}{x^2}$).

Prove that $PV'_{\frac{1}{x}} = -FP_{\frac{1}{x^2}}$.

The following result shows that, at least locally, a distribution is always a derivative (of order mn, in the sense of distributions) of a distribution associate to a bounded function.

Theorem 36 ("Structure locale d'une distribution" Th. XXI of [12])**.** *Let $T \in \mathcal{D}'(\Omega)$. Let ω be an open set in Ω such that $\overline{\omega}$ is a compact in Ω (i. e. ω is a relatively compact open subset of Ω).*

Then there exist $m \in \mathbb{N}$ and $f \in L^\infty(\omega)$ such that

$$T = D_1^m D_2^m \dots D_n^m T_f \quad in \ \ \omega,$$

i. e., for all $\varphi \in \mathcal{D}(\omega)$,

$$T(\varphi) = (-1)^{mn} T_f(D_1^m D_2^m \dots D_n^m \varphi) = (-1)^{mn} \int_\omega f(x) D_1^m D_2^m \dots D_n^m \varphi(x)\,dx.$$

Proof. Suppose that

$$T(\varphi) = (-1)^{mn} \int_\omega f(x) D_1^m D_2^m \dots D_n^m \varphi(x)\, dx \quad \text{for all} \;\; \varphi \in \mathcal{D}(\omega). \tag{16}$$

Consequently

$$|T(\varphi)| \le \|f\|_{L^\infty(\omega)} \int_\omega |D_1^m D_2^m \dots D_n^m \varphi(x)|\, dx$$

i. e. there exists $C > 0$ such that

$$|T(\varphi)| \le C \int_\omega |D_1^m D_2^m \dots D_n^m \varphi(x)|\, dx \quad \text{for all} \;\; \varphi \in \mathcal{D}(\omega). \tag{17}$$

We prove now that (17) implies (16). In fact, suppose (17) holds. Let's define

$$V = \{D_1^m \dots D_n^m \varphi \;\big|\; \varphi \in \mathcal{D}(\omega)\}$$

and consider the functional

$$V \to \mathbb{C}, \qquad D_1^m \dots D_n^m \varphi \mapsto T(\varphi).$$

Thinking at V as a subspace of $L^1(\omega)$, we have that the above functional is linear and moreover condition (17) implies that it is continuous with respect to the norm of $L^1(\omega)$. We use now Hahn-Banach theorem. There exists $\Phi \in (L^1(\omega))'$ such that

$$\Phi(D_1^m \dots D_n^m \varphi) = T(\varphi) \qquad \text{for all} \;\; \varphi \in \mathcal{D}(\omega).$$

From Riesz's theorem we have that there exists $g \in L^\infty(\omega)$ such that

$$\Phi(v) = \int_\omega gv \qquad \text{for all} \;\; v \in L^1(\omega).$$

Consequently

$$T(\varphi) = \Phi(D_1^m \dots D_n^m \varphi) = \int_\omega g(x) D_1^m \dots D_n^m \varphi(x)\, dx \quad \text{for all } v \in L^1(\omega).$$

Taking $f = (-1)^{mn} g$, we have (16).

To conclude the proof it is sufficient to show (17). T is s distribution, then, in particular, there exist $C_{\overline{\omega}} > 0$ and $m_{\overline{\omega}} \in \mathbb{N}$ such that

$$|T(\varphi)| \le C_{\overline{\omega}} \sum_{|\alpha| \le m_{\overline{\omega}}} \sup_\Omega |D^\alpha \varphi| \quad \text{for all } \varphi \in \mathcal{D}(\Omega) \;\; \text{with} \;\; \operatorname{Supp} \varphi \subseteq \overline{\omega}.$$

and consequently there exist $C > 0$ and $m \in \mathbb{N}$, such that

$$|T(\varphi)| \le C \sum_{|\alpha| \le m} \sup_\omega |D^\alpha \varphi| \quad \text{for all } \varphi \in \mathcal{D}(\omega).$$

Since ω is relatively compact, there exists $a > 0$ such that the diameter of ω is less or equal than a. Consequently, if $\psi \in \mathcal{D}(\omega)$, then

$$D^\alpha \psi(x) = \int_{-\infty}^{x_1} \partial_{x_1}(D^\alpha \psi(t, x')\, dt$$

Ex. rem. consider $R \setminus \{0\} \to \mathbb{R}$, $x \longmapsto \frac{1}{x}$ this function is not in $L^1_{loc}(\mathbb{R})$

consider, for $\varphi \in C_c^\infty(\mathbb{R})$,

$$PV_{\frac{1}{x}}(\varphi) := \lim_{\varepsilon \to 0^+} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,dx$$

if this was not ∞: $T_{\frac{1}{x}}(\varphi) = \int_{\mathbb{R}} \frac{\varphi(x)}{x}\,dx$

1) prove that this limit exists for all $\varphi \in \mathcal{D}(\mathbb{R})$
2) prove that $PV_{\frac{1}{x}}$ is a distribution (of order 1)
3) prove that $PV_{\frac{1}{x}}$ is $T_{\log|x|}'$

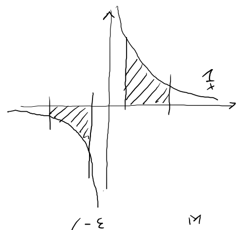
1) given $\varphi \in \mathcal{D}(\mathbb{R})$ prove that there exists

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,dx$$

$\varphi \in \mathcal{D}(\mathbb{R}) \Rightarrow \exists M>0 \ \text{supp}\,\varphi \subseteq [-M, M]$

$$= \int_{-M}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx + \int_{\varepsilon}^{M} \frac{\varphi(x)}{x}\,dx$$

remark $\int_{-M}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{M} \frac{1}{x}\,dx = 0$



so that

$$= \int_{-M}^{-\varepsilon} \frac{\varphi(x)}{x} - \frac{\varphi(0)}{x}\,dx + \int_{\varepsilon}^{M} \frac{\varphi(x)}{x} - \frac{\varphi(0)}{x}\,dx$$

$$\left(\int_{-M}^{-\varepsilon} \frac{\varphi(0)}{x}\,dx + \int_{\varepsilon}^{M} \frac{\varphi(0)}{x}\,dx = 0 \right)$$

$$= \int_{-M}^{-\varepsilon} \underbrace{\frac{\varphi(x) - \varphi(0)}{x}}_{\psi(x)}\,dx + \int_{\varepsilon}^{M} \underbrace{\frac{\varphi(x) - \varphi(0)}{x}}_{\psi(x)}\,dx$$

$$\psi(x) = \begin{cases} \frac{\varphi(x) - \varphi(0)}{x} & \text{if } x \neq 0 \\ \varphi'(0) & \text{if } x = 0 \end{cases}$$

remark that ψ is continuous

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,dx = \lim_{\varepsilon \to 0^+} \int_{[-M,M] \setminus [-\varepsilon, \varepsilon]} \psi(t)\,dt = \boxed{\int_{-M}^{M} \psi(t)\,dt}$$

so $PV_{\frac{1}{x}}(\varphi)$ exists and it is equal to $\int_{-M}^{M} \psi(t)\,dt$ where $\psi(t) = \begin{cases} \frac{\varphi(t) - \varphi(0)}{t} & t \neq 0 \\ \varphi'(0) & t = 0 \end{cases}$ ($= \varphi'(\theta)$, $0 < \theta < t$)

(point 1)

the 2) $|PV_{\frac{1}{x}}(\varphi)| \leq \int_{-M}^{M} |\psi(t)|\,dt$

$|\psi(t)| \leq \sup |\varphi'|$ first derivative $\Rightarrow$ the order ≤ 1

$$\boxed{|PV_{\frac{1}{x}}(\varphi)| \leq 2M \sup_{x \in \mathbb{R}} |\varphi'(x)|} \Rightarrow PV_{\frac{1}{x}} \in \mathcal{D}'(\mathbb{R})$$

this constant depend on $[-M, M]$ the compact set

3) prove that $PV_{\frac{1}{x}} = (T_{\log|x|})'$ (d)

$$T_{\log|x|}(\varphi) = \int_{-\infty}^{+\infty} \log|x| \cdot \varphi(x)\,dx$$

$$(T_{\log|x|})'(\varphi) = -T_{\log|x|}(\varphi') = -\int_{-\infty}^{+\infty} \underbrace{\log|x| \cdot \varphi'(x)}_{\in L^1(\mathbb{R})}\,dx$$

$$= \lim_{\varepsilon \to 0^+} -\int_{-\infty}^{-\varepsilon} \log|x| \cdot \varphi'(x)\,dx - \int_{\varepsilon}^{+\infty} \log|x|\, \varphi'(x)\,dx$$

$$-\int_{-\infty}^{-\varepsilon} \log(-x)\, \varphi'(x)\,dx = -\log(-x) \cdot \varphi(x) \Big|_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx$$

$$-\int_{\varepsilon}^{+\infty} \log(x)\, \varphi'(x)\,dx = -\log(x) \cdot \varphi(x) \Big|_{\varepsilon}^{+\infty} + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,dx$$

$$\lim_{\varepsilon \to 0^+} \left(-\left(\int_{-\infty}^{-\varepsilon} \log(-x)\varphi'(x)\,dx + \int_{\varepsilon}^{+\infty} \log(x)\varphi'(x)\,dx \right) \right) = -\log(\varepsilon)\varphi(-\varepsilon) + \log(\varepsilon)\varphi(\varepsilon) + \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,dx$$

$$= \lim_{\varepsilon \to 0^+} \left(\log(\varepsilon)\left(-\varphi(-\varepsilon) + \varphi(\varepsilon)\right) \right) + \underbrace{\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx + \int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x}\,dx}_{PV_{\frac{1}{x}}(\varphi)}$$

$$\lim_{\varepsilon \to 0^+} \underbrace{\varepsilon \log \varepsilon}_{\to 0} \underbrace{\left(\frac{-\varphi(-\varepsilon) + \varphi(\varepsilon)}{\varepsilon} \right)}_{2\varphi'(0)} = 0$$

OK