Thm. K, A integral domain finitely generated K-algebra $A = K[z_1, \dots, z_n]$

1) $\dim A = \text{tr.d.}\, Q(A)/K$

$= \text{tr.d.}\, K(z_1, \dots, z_n)/K \le n$

2) $P \subseteq A$ prime ideal: $\dim A = \dim A/P + \text{ht} P$.

$R \subseteq T$ R subring of the ring T

$Q \subset T$ prime ideal, $P = Q \cap R$ is a prime ideal in R

$a, b \in R$ $ab \in P = Q \cap R \subset Q \implies a \in Q \implies a \in P$ / $b \in Q \implies b \in P$

$P = Q \cap R$ contraction of Q in R

$P \subset R$ prime ideal: $\exists ?$ a prime ideal Q in T s.t.

$P = Q \cap R$? 4 properties that could hold for $R \subseteq T$.

(LO) = Lying over: $\forall P \subset R$ prime $\exists Q$ prime in T s.t.

$Q \cap R = P$ Q is over P

(GU) = Going-up $P \subsetneq P_0$ prime ideals in R

$Q \subset Q_0$ $Q \cap R = P$

$\exists Q_0$ prime in T s.t. $Q_0 \cap R = P_0$, $Q \subset Q_0$

(GD) = Going-down $P \subsetneq P_0$ prime in R

Q_0 prime in T $Q_0 \cap R = P_0$

$\exists Q$ prime in T s.t. $Q \cap R = P$, $Q \subset Q_0$

(INC) = Incomparable: if Q, Q_0 are prime in T

s.t. $Q \cap R = Q_0 \cap R \implies Q, Q_0$ are incomparable.

$Q \not\subseteq Q_0$ and $Q_0 \not\subseteq Q$.

If $Q \subsetneq Q_0$ prime in $T \iff Q \cap R \subsetneq Q_0 \cap R$

Thm. of Cohen-Seidenberg say that these prop.

hold under some assumption on $R \subseteq T$.

Prop. $R \subseteq T$ $\quad \overline{R} = \{ x \in T \mid x \text{ is integral over } R \}$

$\Longrightarrow \overline{R}$ is a subring of T.

Pf- $x, y \in \overline{R}$ $\quad R[x,y]$ is a finite R-algebra

$x-y, xy \in R[x,y] \Longrightarrow x-y, xy$ are integral over R

$\Longrightarrow x-y, \underline{xy} \in \overline{R}$ $\quad \Longrightarrow \overline{R}$ is a ring $\quad \square$

$K \subset E$

Def. $\overline{R}$ integral closure of R in T —

R is integrally closed in T if $\overline{R} = R$

Ass. R is an integral domain, take $T = Q(R)$.

$R \subset Q(R)$ $\quad \overline{R}$ integral closure of R

If $R = \overline{R}$: R is called integrally closed &

R is a <u>normal domain</u>.

Prop. Ass. R is a UFD $\Rightarrow$ R is a normal ring.

$R=\mathbb{Z}$

$Q(\mathbb{Z})=\mathbb{Q}$

Pf: $Q=Q(R)$, $x\in Q$ $\quad x=\frac{f}{g}$, $f,g\in R$,
$g\in R$

Ass. x is integral over R:

$$x^n+a_1x^{n-1}+\dots+a_n=0\ ,\ a_1,\dots,a_n\in R$$

$$\left(\frac{f}{g}\right)^n+a_1\left(\frac{f}{g}\right)^{n-1}+\dots+a_n=0\ /g^n$$

Ass. f,g relatively prime

$$f^n=-(a_1f^{n-1}g+\dots+a_ng^n)=\underline{g}\,(-a_1f^{n-1}-\dots-a_ng^{n-1})$$

If p is any prime factor of g $\Rightarrow p\mid f^n\Rightarrow p\mid f$

So $g\mid f$, but $(f,g)=1$ $\Rightarrow$ g is invertible

$\Rightarrow$ $x=\frac{f}{g}\in R$.

Cor: $\mathbb{Z}$, $K[x_1,\dots,x_n]$ are normal rings.

Theorems of Cohen-Seidenberg: $R \subseteq T$,

1) Ass. that T is integral extension of R then:
(LO), (GU), (INC) are all satisfied.

2) Ass. R, T are integral domains, T is integr. ext of R and $\underline{R \text{ is normal}}$ Then (GD) is satisfied.

$\underline{\text{Theorem}}$ $\underline{K}$, A finitely generated K-algebra,
Ass. A is integral extension of $K[\underbrace{z_1, \dots, z_n}_{\text{alg. indep. over } K}]$.

Then:

1) $\underbrace{P_0 \subsetneq} P_1 \subset \dots \subset P_\ell$ chain of prime ideals of A of length ℓ

$\ell \leq n$

2) If the chain is not extendable $\Rightarrow$
$\ell = n$ if and only if $P_0 \cap K[z_1, \dots, z_n] = (0)$.

Pf Induction on n.

$n=0$ $\quad K[z_1, \dots z_n] = K$: A is integral exten. of K

We want to prove that any chain of prime ideals has length 0 i.e. if P prime ideal then P is maximal.

$K \hookrightarrow A \xrightarrow{\pi} A/P$: this composition has kernel $P \cap K = (0)$ so is injective

A/P is a K-algebra integral extension of K: $a \in A$

$[a]$ $\quad$ a integral over K: $a^r + c_1 a^{r-1} + \dots + c_r = 0$, $c_1, \dots, c_r \in K$

$[a]^r + \underbrace{[c_1]}_{= c_1}[a]^{r-1} + \dots + \underbrace{[c_r]}_{= c_r} = 0$: $[a]$ is integral over K with same equation of integral dep. as a

$\underbrace{A/P}$ is integral extension of K, field $\Longrightarrow$ P is maximal

Claim $\quad K \subseteq T$ ring (domain) integral ext. of $K \Rightarrow T$ is a field.

$t \in T$, $t \neq 0$: $\underbrace{\left(t^r + c_1 t^{r-1} + \dots + c_r = 0 \right)}$, $c_1, \dots, c_r \in K$

We can ass. r is the min degree of an equal. of integ. dep. for t over K $\Longrightarrow c_r \neq 0$: otherwise

$c_r = 0$ $\quad t^r + c_1 t^{r-1} + \dots + c_{r-1} t =$
$$= t\underbrace{(t^{r-1} + c_1 t^{r-2} + \dots + c_{r-1})}_{\deg < r} = 0$$ in T integ. dom., $t \neq 0$

$\deg < r$: contrad.

$c_r^{-1} \in K$ $\quad \underbrace{t^r c_r^{-1} + \dots + c_{r-1} t c_r^{-1}} + 1 = 0$

$$\underline{1} = -\underline{t}\underbrace{(t^{r-1} c_r^{-1} + \dots + c_{r-1} c_r^{-1})}_{\text{this is } t^{-1}}$$

$\ell = n = 0$ $\quad P_0 \cap K = (0)$

Inductive step: ass. $m > 0$, ass. thm true for $m-1$

$P_0 \subset P_1 \subset \cdots \subset P_\ell$ in A

Idea to use the inductive ass. is to apply it to $A/(f)$;

A is integral over $K[z_1 \ldots z_n]$

$P_1 \cap K[z_1 \ldots z_n] = Q_1$ $\quad Q_0 \subset \underbrace{Q_1 \subset \cdots \subset Q_\ell}$ in $K[z_1 \ldots z_n]$

chain of length ℓ (INC)

If $\ell = 0 \Rightarrow \ell < m$ ok.

Ass. $\ell > 0$ $\quad Q_1 \neq (0) \Rightarrow \exists f \in Q_1,\ f \neq 0,\ f$ irred.

$$\boxed{\frac{Q_1}{(f)} \subset \frac{Q_2}{(f)} \subset \cdots \subset \frac{Q_\ell}{(f)}}$$ chain in $\frac{K[z_1 \ldots z_n]}{(f)} = B$ int. dom. of length $\ell - 1$

$B = \frac{K[z_1 \ldots z_n]}{(f)}$ integ. dom. K-algebra. Normalization Lemma $\Rightarrow$

it is integ. ext. of $K[\underbrace{y_1, \ldots y_{m-1}}_{\text{alg. indep.}}]$ $\quad r = \text{tr.d.}\, Q(B)/K$

$r \quad f(z_1, \ldots, z_n) = 0$: unique equat. of alg. dep.

$r = m-1$

Apply induction to $B \Rightarrow \ell - 1 \leq m-1$

$\boxed{\ell \leq m}$

2) $P_0 \subset P_1 \subset \cdots \subset P_\ell$ unextendible

claim $\ell = m \iff P_0 \cap K[z_1 \ldots z_n] = (0)$

$P_0 \subset \cdots \subset P_\ell$ max.-ext.

$Q_0 = P_0 \cap K[z_1 \ldots z_n] \neq (0) \Longrightarrow \exists\, g \in Q_0,\ g \neq 0$

$\dfrac{K[z_1 \ldots z_n]}{(g)}$ is integral over $K[\underbrace{y_1 \ldots y_{n-1}}_{\text{alg. indep.}}]$

$$\underbrace{\frac{Q_0}{(g)} \subset \frac{Q_1}{(g)} \subset \cdots \subset \frac{Q_\ell}{(g)}}_{\leq n-1} \quad \text{length } \ell$$

$\Longrightarrow \ell \leq n-1 \qquad \boxed{\ell < n}.$

$\boxed{P_0 \subset \cdots \subset P_\ell}$ Ass. $Q_0 = P_0 \cap K[z_1 \ldots z_n] = (0)$

$K[z_1 \ldots z_n] \longrightarrow A \xrightarrow{\pi} A/P_0$: the composition is injective

A/P_0 can be interpreted as an algebra over $K[z_1 \ldots z_n]$, integral extension of $K[z_1 \ldots z_n]$

(a) : we can reduce mod P_0 an eq. of integr. dep. for a over $K[z_1 \ldots z_n]$

A/P_0 integr. dom., $K[z_1, \ldots, z_n]$ normal

$K[z_1 \ldots z_n] \subseteq A/P_0$: (GD) holds

$Q_1 \ni f \neq 0$ irred. $(f) \subseteq Q_1$ prime ideals

$$Q_1 = P_1 \cap K[z_1 \ldots z_n]$$

$P_1/P_0 \cap K[z_1 \ldots z_n] = Q_1$ $P_1/P_0 \subset A/P_0$

(GD) $\Rightarrow \exists\ N$ prime in A/P_0 over (f)

$N \cap K[z_1 \ldots z_n] = (f) \neq 0$

Also the chain $P_0/P_0 \subset P_1/P_0$ is unextendable

$P_0/P_0 = (0)$

$\Rightarrow N = P_1/P_0$, $N \cap K[z_1 \ldots z_n] = (f) \Rightarrow Q_1 = (f)$

$\dfrac{K[z_1 \ldots z_n]}{(f)} \longrightarrow \dfrac{A}{P_1}$ injective $\Rightarrow$ A/P_1 integral over $\dfrac{K[z_1 \ldots z_n]}{(f)}$

$(0) = P_1/P_1 \subset P_2/P_1 \subset \cdots \subset P_\ell/P_1$ non-ext chain

Induct. $\dfrac{K[z_1 \ldots z_n]}{(f)}$ is integral over $K[y_1 \ldots y_{n-1}]$

the length is equal to $n-1 = \ell - 1$

$\Rightarrow \ell = n$

Corollaries A integral domain, finitely gen. K-algebra, $n = \mathrm{tr.d}\, Q(A)/K$

1) all non-extendable chains of prime ideals in A have length n

Pf. By normaliz. lemma, A is integr. ext of $K[z_1, \dots, z_n]$: we can apply thm

$\underbrace{P_0}_{\substack{\| \\ (0)}} \subset \dots \subset P$ $\quad$ $\cancel{\ell} \leq n$

(0) because (0) is prime in A

$\Rightarrow P_0 \cap K[z_1, \dots z_n] = (0)$: $\ell = n$

2) $\dim A = n$

3) $Q \subset P$ prime in A

$Q = P_0 \subset P_1 \subset \dots \subset P_\ell = P$ chain

$\Rightarrow$ the length of this chain is

$\ell = \mathrm{tr.d}\, Q(A/Q)/K - \mathrm{tr.d.}\, Q(A/P)/K$

$(0) \subset \dots \subset Q = P_0 \subset \dots \subset P_\ell = P \subset P_{\ell+1} \subset \dots$

chain of length n

$Q/Q \subset \dots \subset P_\ell/Q \subset P_{\ell+1}/Q \subset \dots$

non-ext. in A/Q : its length is $\dim A/Q$ applying thm to A/Q integr. dom., fin. gen. K-alg.

$P_\ell/P \subset P_{\ell+1}/P \subset \dots$: length $\dim A/P$

$\dim A/Q - \dim A/P = \mathrm{tr.d.}\, Q(A/Q)/K - \mathrm{tr.d.}\, Q(A/P)/K$

4) $M \subset A$ maximal $\Rightarrow$ ht $M = \dim A$

5) $P \subset K[x_1, \dots x_n]$ prime

$\dim A/P = n - \mathrm{ht}\, P$ $\leftarrow$

Pf. $\underbrace{(0) = P_0 \subset \dots \subset}_{\mathrm{ht}\, P} \underbrace{\underset{\substack{\| \\ P_\ell}}{P} \subset \dots \subset P_n}_{\dim A/P}$

$\exists$ examples of noeth. domains with non-extendable chains having different lengths, and also of infinite Krull dimension.

Catenary ring Matsumura