$X \subseteq \mathbb{P}^n_K$ locally closed in Zariski topology
$\overset{\shortparallel}{Z \cap U}$. If irreducible, X is quasi-projective var.

Z closed, U open

A <u>function</u> is a $\varphi : X \longrightarrow K$

<u>Def.</u> $\varphi : X \longrightarrow K$ is a regular function on X if
$\forall P \in X$ $\exists$ U_P open neighbourhood of P s.t.
$\varphi|_{U_P} = \frac{F}{G}$, $F, G \in K[x_0, \dots, x_n]_d$ homogeneous of the same degree d

Def of <u>local character</u>.

$F \in K[x_0 \dots x_n]$, $P[a_0, \dots, a_n]$ $F(a_0, \dots, a_n) \neq F(\lambda a_0, \dots, \lambda a_n)$

$F(\lambda a_0, \dots, \lambda a_n) = \lambda^d F(a_0, \dots, a_n)$ F homog of deg d

$F, G \in K[x_0 \dots x_n]_d$

$$\frac{F(\lambda a_0, \dots, \lambda a_n)}{G(\lambda a_0, \dots, \lambda a_n)} = \frac{\cancel{\lambda^d} F(a_0, \dots, a_n)}{\cancel{\lambda^d} G(a_0, \dots, a_n)}$$ independent of λ

If $G(a_0, \dots, a_n) \neq 0 \Longrightarrow \frac{F}{G}$ defines a function on $\mathbb{P}^n \setminus V_P(G)$

$\mathcal{O}(X) = \{\varphi : X \to K \mid \varphi \text{ is regular}\} \supseteq K$

$c \in K \rightsquigarrow$ constant function $c : X \to K$ $\quad c = \frac{c}{1}$

$\boxed{\mathcal{O}(X) \text{ is a } K\text{-algebra}}$ defining:

$\varphi + \psi$ where $\varphi, \psi \in \mathcal{O}(X)$ $\quad (\varphi+\psi)(P) = \varphi(P) + \psi(P)$

$\varphi\psi$ $\quad (\varphi\psi)(P) = \varphi(P)\psi(P)$

$\varphi + \psi$, $\varphi\psi$ are regular functions on X

$P \in X$ $\quad \exists U_P$ $\quad \varphi|_{U_P} = \frac{F}{G}$ $\quad$ deg d $\quad$ $G \neq 0$ at any pt of U_P

$\exists U'_P$ $\quad \psi|_{U'_P} = \frac{F'}{G'}$ $\quad$ deg d' $\quad$ $G' \neq 0 \ldots U'_P$

$U_P \cap U'_P$ $\quad \varphi + \psi|_{\underline{U_P \cap U'_P}} = \frac{F}{G} + \frac{F'}{G'} = \frac{FG' + F'G}{\underline{GG'}}$ $\quad$ homog. of deg $d+d'$

$\varphi\psi|_{U_P \cap U'_P} = \frac{FF'}{GG'}$

$X \rightsquigarrow \mathcal{O}(X)$ ring of regular functions on X

Prop. $X \subseteq \mathbb{A}^n_K$ $\quad \varphi : X \longrightarrow K$ regular on X

$K = \mathbb{A}^1_K$ Zariski top.

$\Longrightarrow \varphi$ is continuous.

Pf. $Z \subseteq \mathbb{A}^1_K$ Z closed : we have to check that $\varphi^{-1}(Z)$ closed in X

$Z = \mathbb{A}^1$ ✓ $\qquad Z$ finite set

$Z = \emptyset$ ✓

Claim $\forall c \in K$, $\varphi^{-1}(c)$ is closed

$$\varphi^{-1}(c) = \{ P \in X \mid \varphi(P) = c \}$$

Lemma T top space, $T = \bigcup_{i \in I} U_i$ open covering

$Z \subseteq T$ $\quad Z$ is closed $\Longleftrightarrow$ $Z \cap U_i$ is closed in U_i $\forall i \in I$

"Being closed in a top space is a local property"

Ass. theorem is true, $c \in K$

$\varphi^{-1}(c) = \{ P \in X \mid \varphi(P) = c \}$

$\forall P \in X \quad \exists\, U_P \quad \varphi|_{U_P} = \frac{F_P}{G_P}$, F_P, G_P homog. of same degree

$X = \bigcup_{P \in X} U_P$ open covering

$\varphi^{-1}(c)$ is closed $\iff$ $\underline{\varphi^{-1}(c) \cap U_P}$ is closed in U_P $\forall P$

$\varphi^{-1}(c) \cap U_P = \left\{ x \in U_P \,\middle|\, \underline{\frac{F_P(x)}{G_P(x)} = c} \right\} =$

$= \{ x \in U_P \mid \underbrace{(F_P - c\,G_P)}_{\text{homog. pol.}}(x) = 0 \} =$

$= \underline{U_P \cap V_P(F_P - c\,G_P)}$ closed in U_P

Proof of the Lemma. $T = \bigcup_{i \in I} U_i$. $\quad C_i = T \setminus U_i$ closed in T

Ass. $Z \cap U_i$ is closed in U_i $\forall i$: $\exists Z_i$ closed in T s.t.
$Z \cap U_i = Z_i \cap U_i$

Claim $Z \overset{\subseteq}{\underset{\supseteq}{=}} \bigcap_{i \in I} (Z_i \cup C_i)$

If $P \in Z \Rightarrow \exists \bar{i}$ $P \in Z \cap U_{\bar{i}} = Z_{\bar{i}} \cap U_{\bar{i}}$ $\quad P \in Z_{\bar{i}}$

If for some j $P \notin Z \cap U_j$, $P \notin Z_j$, $P \notin U_j \Rightarrow P \in C_j$.

For any index $i \Rightarrow P \in Z_i \cup C_i \Rightarrow P \in \bigcap (Z_i \cap C_i)$

conversely: $P \in \bigcap_{i \in I} (Z_i \cup C_i)$: $\forall i$ $P \in Z_i$ or $P \in C_i$.

$\exists \bar{i}$ $P \in U_{\bar{i}} \Rightarrow P \notin C_{\bar{i}} \Rightarrow P \in Z_{\bar{i}} \Rightarrow P \in Z$

$\varphi \in \mathcal{O}(X) \Rightarrow \varphi : X \to \mathbb{A}^1$ continuous

Consequence 1) $0 \in K$ $\varphi^{-1}(0)$ is closed in X

$\varphi \in \mathcal{O}(X)$: the set of zeros of φ is closed in X

$V(\varphi) = \{P \in X \mid \varphi(P) = 0\}$ relative situation

2) X quasi-projective variety $\boxed{\text{IRREDUCIBLE}}$

$\varphi, \psi \in \mathcal{O}(X)$: if φ, ψ coincide on $U \neq \emptyset$ open in X

$\Longrightarrow \varphi = \psi$ in $\mathcal{O}(X)$.

Pf $\varphi - \psi \in \mathcal{O}(X)$ $V(\varphi - \psi)$ closed in X

$$V(\varphi-\psi) = \{P \in X \mid \varphi(P) = \psi(P)\}$$

$V(\varphi - \psi) \supseteq U$ open dense in X $\Longrightarrow$

$V(\varphi - \psi) \supseteq \overline{U} = X$ $\Longrightarrow \boxed{V(\varphi - \psi) = X}$

$X \subset \mathbb{P}^n_K$ locally closed ; $X \subseteq \mathbb{A}^n_K$

$\cup|$

$U_0 = \mathbb{A}^n_K$

$P = (a_1, \dots, a_n) = [x_0, \dots, x_n] = [1, a_1, \dots, a_n]$

$\frac{x_1}{x_0} = a_1, \dots, \frac{x_n}{x_0} = a_n$

$\varphi \in \mathcal{O}(X)$ $\quad P \in X \quad U_P \quad \varphi|_{U_P} = \frac{F}{G}$, F, G homog. of deg d

~~we are~~ using homog. coordinates on X

$$\varphi(P) = \frac{F(1, a_1, \dots, a_n)}{G(1, a_1, \dots, a_n)} = \frac{{}^aF}{{}^aG}$$

on U_P

expression in non-homog. coordinates

polynomials in $K[x_1, \dots, x_n]$

not required homog. of the same degree

If $\varphi : X \longrightarrow K$, $\forall P$ $\exists U_P$, $\exists$ $A, B \in K[x_1 \dots x_n]$ st.

$\mathbin{\rotatebox[origin=c]{90}{$\subseteq$}}$

$\mathbb{A}^n$

$\varphi|_{U_P} = \frac{A}{B}$ in non-homog. coord.

$$\varphi(a_1 \dots a_n) = \frac{A(a_1 \dots a_n)}{B(a_1 \dots a_n)} = \frac{{}^hA(x_0, \dots, x_n)}{{}^hB(x_0, \dots, x_n)} \qquad r = \deg B - \deg A$$

x_0^r

$[x_0 \dots x_n]$

$$A \longrightarrow x_0^r\, {}^hA = x_0^r x_0^{\deg A} A\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

$${}^hB = x_0^{\deg B} B\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

$\deg B \geq \deg A$

homog of same degree

$\frac{x_0^r\, {}^hA}{{}^hB}$ is the expression of φ on U_P in homog. coord.

If $X \subseteq \mathbb{A}^n$, $\varphi \in \mathcal{O}(X)$ is a function locally expressed by quotients of pol. in $K[x_1 \dots x_n]$

$X \subseteq \mathbb{A}^n$ closed $\quad K[X]$ coordinate ring
U_0

$K[X] = \{ f : X \to K \mid f \text{ defined by a pol. } F \}$

f is a regular function on X

$$\boxed{f = \frac{{}^{a}F}{x_0^{\deg F}}} \qquad f(a_1, \dots, a_n) = \left(\frac{x_0^{\deg F} F\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)}{x_0^{\deg F}} \right)(a_1 \dots a_n)$$

expression on the whole X : $\forall\, p \in X \quad U_p = X$

$\underline{K[X]} \subseteq \underline{\mathcal{O}(X)}$ sub-K-algebra

$\alpha \subseteq K[X]$ ideal $\qquad V(\alpha) \subseteq X$ closed in X

$$V(\alpha) = \bigcap_{\varphi \in \alpha} \underbrace{V(\varphi)}_{\text{closed}}$$

$\alpha = \dfrac{I}{I(X)} \subseteq \dfrac{K[x_1 \dots x_n]}{I(X)}$ where $I \subset K[x_1 \dots x_n]$, $\underline{I \supseteq I(X)}$

$$\boxed{V(\alpha) = V(I)} \qquad V(I) \subseteq V(I(X)) = X$$

Relative Nullstellensatz for X affine variety

If K is algebraically closed, Nullstellensatz implies that: $\alpha \subsetneq K[X] \Rightarrow \alpha = \dfrac{I}{I(X)}$, with $I \subsetneq K[x_1 \dots x_n]$ therefore $\boxed{V(\alpha) \neq \emptyset}$

$f \in K[X]$, if f vanishes on $V(\alpha) = V(I)$

$\Rightarrow F \in \sqrt{I} \Rightarrow f \in \sqrt{\alpha}$

If f vanishes on $V(g_1, \dots, g_m)$, $g_1, \dots, g_m \in K[X]$, then $f \in \sqrt{(g_1, \dots, g_m)}$

$X \subseteq \mathbb{A}^n_K$ closed $\quad K[X] \subseteq \mathcal{O}(X)$

Thm. If K is algebraically closed, then $K[X] = \mathcal{O}(X)$: every regular function on X is a polynom. function.

Proof. $\varphi \in \mathcal{O}(X)$ $\quad$ We want to prove that $\varphi \in K[X]$

First case : assume $\boxed{X \text{ irreducible.}}$

$\forall P \in X \quad U_P, F_P, G_P \quad \varphi|_{U_P} = \frac{F_P}{G_P}, \quad V(G_P) \cap U_P = \emptyset$

$F_P, G_P \in K[x_1, \dots x_n]$ $\quad$ expression in non-homog. coord.

$\forall F_P, G_P \ \forall P \rightsquigarrow f_P, g_P \in K[X] \subseteq \mathcal{O}(X), \ \varphi \in \mathcal{O}(X)$

$\boxed{\varphi|_{U_P} = \frac{f_P}{g_P}}$ on U_P $\quad g_P \varphi - f_P \in \mathcal{O}(X)$

it vanishes on $U_P \subseteq X$ $\quad$ X IRREDUCIBLE

$\Longrightarrow \boxed{g_P \varphi - f_P = 0 \text{ in } \mathcal{O}(X)}$

$\langle \{ g_P \mid P \in X \} \rangle = \alpha \subseteq K[X]$

$V(\alpha) = \emptyset$: $\quad \forall P \in X \quad g_P(P) \neq 0$

$\alpha = K[X] \ni 1 \quad$ 1 is a combination of the generators $\{g_P\}_{P \in X}$

$1 = \sum_{\text{finite}} h_P g_P$, $h_P \in K[X]$

in $K[X] \subseteq \mathcal{O}(X)$

$\varphi = \varphi \sum_{\text{fin}} h_P g_P =$

$= \sum_{\text{fin}} h_P (g_P \varphi) =$

$= \sum_{\text{fin}} \underbrace{h_P f_P}_{\in K[X]} \in K[X]$

2) X non nec. irreducible

$\varphi|_{U_P} = \frac{f_P}{g_P}$ $f_P, g_P \in K[X]$, $V(g_P) \cap U_P = \emptyset$

$X \setminus U_P$ closed in X $X \setminus U_P \subsetneq X \implies$

$\exists R \in K[x_1 \dots x_n]$ s.t. $V(R) \supseteq X \setminus U_P$

$V(R) \not\supseteq X$

i.e. $I(X \setminus U_P) \supsetneq I(X)$ $r \in \mathcal{O}(X)$

r vanishes at $X \setminus U_P$, $r \neq 0$ in $\mathcal{O}(X)$

$X \setminus V(R) \subseteq U_P$ $X \setminus U_P \not\ni P$

open in X

$\underbrace{(X \setminus U_P) \cup \{P\}}_{\text{closed}} \supsetneq \underbrace{X \setminus U_P}_{\text{closed}}$

We can choose R s.t. $R(P) \neq 0$ so that

$R \in I(X \setminus U_P)$ but $R \notin I((X \setminus U_P) \cup \{P\})$

$r(P) \neq 0$ $X \setminus V(r)$ is an open nbhd of P

$\subseteq U_P$

$\varphi|_{U_P} = \frac{f_P}{g_P} = \boxed{\frac{f_P r}{g_P r}}$ on $X - V(r)$ smaller nbhd of P

$f_P r$, $\varphi g_P r \in \mathcal{O}(X)$ are equal

On U_P $g_P \varphi = f_P$

On $X - U_P$ $r \equiv 0 \longrightarrow \boxed{\varphi g_P r = f_P r}$

$0 = 0$

Same situation as X irreducible $\implies$

same conclusion.

K alg closed, X affine var. $\boxed{\mathcal{O}(X) = K[X]}$

If X is an irreducible projective variety, K alg. closed: $\boxed{\mathcal{O}(X) = K}$

X locally closed set $\quad$ $\mathcal{O}(X)$ is an interesting object

X, $U \subseteq X$ open $\quad \{ \underline{\underline{\mathcal{O}(U)}} \mid U \subseteq X \text{ open} \}$

$\rightsquigarrow$ sheaf of regular functions on X