Chapter 8

Dimension of K-algebras.

The purpose of this chapter is to prove Theorem 7.2.4. In reality we will not give a complete proof of it, but we will only enunciate the Cohen-Seidenberg theorems and then we will see how, from them and from the Normalization Lemma, the theorem follows.

8.1 Prime ideals of integral extensions

Let $R \subset T$ be rings, R subring of T. We are interested in relations between the prime ideals of R and those of T. We are principally concerned with the case where T is integral over R, but we formulate the definitions in greater generality.

It is easily seen that if $\mathcal{Q}$ is a prime ideal of T, then $\mathcal{Q} \cap R$ is a prime ideal of R, called contraction of $\mathcal{Q}$ in R. We list four properties that might hold for a pair $R \subset T$.

(LO) *Lying over.* For any prime ideal $\mathcal{P}$ in R there exists a prime ideal $\mathcal{Q}$ in T with $\mathcal{Q} \cap R = \mathcal{P}$.

(GU) *Going up.* Given prime ideals $\mathcal{P} \subset \mathcal{P}_0$ in R and $\mathcal{Q}$ in T with $\mathcal{Q} \cap R = \mathcal{P}$, there exists a prime ideal $\mathcal{Q}_0$ in T satisfying $\mathcal{Q} \subset \mathcal{Q}_0$ and $\mathcal{Q}_0 \cap R = \mathcal{P}_0$.

(GD) *Going down.* Given prime ideals $\mathcal{P} \subset \mathcal{P}_0$ in R and $\mathcal{Q}_0$ in T with $\mathcal{Q}_0 \cap R = \mathcal{P}_0$, there exists a prime ideal $\mathcal{Q}$ in T satisfying $\mathcal{Q} \subset \mathcal{Q}_0$ and $\mathcal{Q} \cap R = \mathcal{P}$.

(INC) *Incomparable.* Two different prime ideals in T with the same contraction in R cannot be comparable: if $\mathcal{Q} \subsetneq \mathcal{Q}_0$ are prime ideals of T, then $\mathcal{Q} \cap R \subsetneq \mathcal{Q}_0 \cap R$.

Next Theorem 8.1.4 states conditions on the pair of rings that ensure the validity of the above properties. We first need some definitions.

Proposition 8.1.1. *Let $R \subset T$. The set $\overline{R}$ of all elements of T that are integral over R is a subring of T.*

Proof. It relies on Theorem 4.0.1. If $x, y \in \overline{R}$, $R[x, y]$ is a finite R-module. Therefore $x + y, x - y, xy$ are integral over R, because they all belong to $R[x, y]$. □

Definition 8.1.2. $\overline{R}$ is called *the integral closure of R in T*. R is called *integrally closed* in T if $\overline{R} = R$. An integral domain that is integrally closed in its field of fractions is called *normal*.

Next Proposition extends Exercise 4 of Chapter 3.

Proposition 8.1.3. *If A is a UFD, then it is a normal ring. In particular, any polynomial ring with coefficients in a field is normal.*

Proof. Indeed, let $f/g \in Q(A)$ be an element of the quotient field of A, with f, g coprime. Assume that f/g is integral over A: we have an equation of integral dependence

$$(f/g)^r + a_1(f/g)^{r-1} + \cdots + a_r = 0,$$

with coefficients $a_1, \ldots, a_r \in A$. Multiplying everything by g^r we get:

$$f^r = -a_1 f^{r-1} g - \cdots - a_r g^r = g(-a_1 f^{r-1} - \cdots - a_r g^{r-1},$$

therefore $g | f^r$. So each irreducible factor of g divides f. Since f, g are coprime, we conclude that $g = \pm 1$ and $f/g \in A$. □

Theorem 8.1.4. *Let $R \subset T$ be rings with T integral over R. Then:*

1. *the pair $R \subset T$ satisfies (LO), (INC) and (GU);*
2. *if moreover R and T are integral domains and R is normal, then also (GD) is satisfied.*

Proof. For a proof, see for instance [AM] or [P]. □

8.2 Length of chains of prime ideals in K-algebras

Next Theorem 8.2.2 is the key to prove Theorem 7.2.4. First we need to state one more property of integral extensions.

Proposition 8.2.1. *Let $R \subset T$ be integral domains, T integral over R. Then T is a field if and only if R is a field.*

Proof. Suppose R is a field, let $y \in T, y \neq 0$. Let

$$y^n + r_1 y^{n-1} + \cdots + r_n = 0, \ r_i \in R$$

be an equation of integral dependence for y of smallest possible degree. Since T is an integral domain we have $r_n \neq 0$, so $y^{-1} = -r_n^{-1}(y^{n-1} + r_1 y^{n-2} + \cdots + r_{n-1}) \in T$. Hence T is a field.

Conversely, suppose that T is a field; let $x \in R, \ x \neq 0$. Then $x^{-1} \in T$, so it is integral over R, so that we have an equation

$$x^{-m} + s_1 x^{-m+1} + \cdots + s_m = 0, \ s_i \in R.$$

It follows that $x^{-1} = -(s_1 + s_2 x + \cdots + s_m x^{m-1}) \in R$, therefore R is a field. □

Theorem 8.2.2. *Let K be a field, let A be a finitely generated K-algebra, integral extension of $K[z_1, \ldots, z_n]$, with $z_1, \ldots, z_n$ algebraically independent over K. Then:*

a) Every chain of prime ideals of A: $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ has length $l \leq n$;

b) Assume that the chain is non-extendable, then $l = n$ if and only if

$$\mathcal{P}_0 \cap K[z_1, \ldots, z_n] = (0).$$

Proof. By induction on n.

If $n = 0$, then A is integral extension of K. We claim that every prime ideal $\mathcal{P}$ of A is maximal; indeed, first observe that also $A/\mathcal{P}$ is integral extension of K, because, if $a \in A$, from an equation of algebraic dependence for a over K, passing to the quotient we get a similar equation for $[a]$ over K. So by Proposition 8.2.1 it follows that $A/\mathcal{P}$ is a field, and we conclude that $\mathcal{P}$ is maximal. So $l = 0$. Moreover $\mathcal{P} \cap K = (0)$.

Let $n \geq 1$, and let $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ be a chain of prime ideals in A. Let $\mathcal{Q}_i = \mathcal{P}_i \cap K[z_1, \ldots, z_n]$. Then, by Theorem 8.1.4, (INC), $\mathcal{Q}_0 \subset \cdots \subset \mathcal{Q}_l$ is a chain of prime ideals in $K[z_1, \ldots, z_n]$. If $l = 0$ we are done, so assume $l \geq 1$. Then $\mathcal{Q}_1$ contains a non-zero element, and, since $\mathcal{Q}_1$ is prime and $K[z_1, \ldots, z_n]$ is a UFD, there exists $f \in \mathcal{Q}_1$ irreducible. So we have a chain of length $l - 1$ in $K[z_1, \ldots, z_n]/(f)$, which is an integral domain:

$$\mathcal{Q}_1/(f) \subset \cdots \subset \mathcal{Q}_l/(f).$$

By the Normalization Lemma, $K[z_1, \ldots, z_n]/(f)$ is an integral extension of a polynomial ring $K[y_1, \ldots, y_{n-1}]$. Hence, by the induction hypothesis, we have $l - 1 \leq n - 1$, i.e. $l \leq n$. This proves part a).

Assume now that the chain $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$ is not extendable. Assume $\mathcal{Q}_0 = \mathcal{P}_0 \cap K[z_1, \ldots, z_n] = (0)$. Let $A' = A/\mathcal{P}_0$, $\mathcal{P}'_i = \mathcal{P}_i/\mathcal{P}_0$ for any i. The composite map $K[z_1, \ldots, z_n] \hookrightarrow A \to A/\mathcal{P}_0$ is injective because $\mathcal{Q}_0 = (0)$, so $A/\mathcal{P}_0$ is integral over $K[z_1, \ldots, z_n]$. We have that $K[z_1, \ldots, z_n]$ is a normal ring (see Proposition 8.1.3). Hence, we can apply Theorem 8.1.4 (GD) to this extension of rings, as follows. We have $\mathcal{Q}_0 \subsetneq \mathcal{Q}_1$. As before there exists $f \in \mathcal{Q}_1$ irreducible, generating a prime ideal with $(f) \subset \mathcal{Q}_1$. We have also $\mathcal{Q}_1 = \mathcal{P}'_1 \cap K[z_1, \ldots, z_n]$, so by (GD) property there exists a prime ideal $\mathcal{N} \subset \mathcal{P}'_1$ of A' such that $\mathcal{N} \cap K[z_1, \ldots, z_n] = (f)$. But the chain $\mathcal{P}'_0 \subset \mathcal{P}'_1$ is not extendable and $\mathcal{P}'_0 = (0)$, hence $\mathcal{N} = \mathcal{P}'_1$, and $(f) = \mathcal{Q}_1$. It follows that $K[z_1, \ldots, z_n]/(f)$ is a subring of $A/\mathcal{P}_1$ (in the sense that the induced map $K[z_1, \ldots, z_n]/(f) \to A/\mathcal{P}_1$ is injective) and this is an integral extension. Again by Normalization Lemma, $K[z_1, \ldots, z_n]/(f)$ is integral over a polynomial ring $K[y_1, \ldots, y_{n-1}]$. Since $(0) = \mathcal{P}_1/\mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l/\mathcal{P}_1$ is a non-extendable chain of prime ideals of $A/\mathcal{P}_1$, such that $(0) \cap K[y_1, \ldots, y_{n-1}] = (0)$, by inductive assumption we conclude that $l - 1 = n - 1$.

If $\mathcal{Q}_0 \neq 0$, let $g \in \mathcal{Q}_0$ non 0. The ring $K[z_1, \ldots, z_n]/(g)$ is integral over a polynomial ring in $n-1$ variables, so the chain $\mathcal{Q}_0/(g) \subset \cdots \subset \mathcal{Q}_l/(g)$ has length at most $n-1$ and $l < n$.

□

8.3 Consequences

The following series of Corollaries of Theorem 8.2.2 proves the desired results and more.

Corollary 8.3.1. *Let A be an integral domain finitely generated as K-algebra. Let $n = tr.d.Q(A)/K$. Then*

1. **all** *non-extendable chains of prime ideals of A have length n.*

2. *The Krull dimension of A is finite and equal to n.*

3. *Let $\mathcal{Q} \subset \mathcal{P}$ be two prime ideals of A. If*

$$\mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P}$$

   is a non-extendable chain of prime ideals between $\mathcal{Q}$ and $\mathcal{P}$, then $l = tr.d.Q(A/\mathcal{Q})/K - tr.d.Q(A/\mathcal{P})/K$.

4. *Every maximal ideal of A has height n.*

Proof. By the Normalization Lemma there exist n algebraically independent elements $z_1, \ldots, z_n \in A$, such that A is integral over $K[z_1, \ldots, z_n]$. Since A is a domain, for any non-extendable chain of prime ideals $\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l$, we have $\mathcal{P}_0 = (0)$, hence $\mathcal{Q}_0 = \mathcal{P}_0 \cap K[z_1, \ldots z_n] = (0)$. The proof of (1) follows by Theorem 8.2.2. (2) follows from (1).

To prove (3), note that, by (1), we can extend $\mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P}$ to a non-extendable chain of prime ideals of A of length n:

$$(0) \subset \cdots \subset \mathcal{Q} = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_l = \mathcal{P} \subset \mathcal{P}_{l+1} \subset \ldots .$$

The part of the chain from $\mathcal{Q}$ up has length equal to $\dim A/\mathcal{Q} = tr.d.Q(A/\mathcal{Q})/K$, because there is a natural bijection between the set of prime ideals of $A/\mathcal{Q}$ and that of prime ideals of A containing $\mathcal{Q}$. Similarly the part from $\mathcal{P}$ up has length equal to $\dim A/\mathcal{P} = tr.d.Q(A/\mathcal{P})/K$. So (3) follows.

(4) follows because the last ideal in a non-extendable chain of prime ideals of A must be a maximal ideal. □

Corollary 8.3.2. *Let $\mathcal{P} \subset K[x_1, \ldots, x_n]$ be a prime ideal of the polynomial ring in n variables. Then* $\dim A/\mathcal{P} = n - ht(\mathcal{P})$.

Proof. Let

$$(0) = \mathcal{P}_0 \subset \cdots \subset \mathcal{P} \subset \cdots \subset \mathcal{P}_n \tag{8.1}$$

be a non-extendable chain of length n of prime ideals in $K[x_1, \ldots, x_n]$ passing through $\mathcal{P}$. The subchain $(0) = \mathcal{P}_0 \subset \cdots \subset \mathcal{P}$ is a non-extendable chain of prime ideals contained in $\mathcal{P}$, so it has length $ht\mathcal{P}$, whereas the subchain $\mathcal{P} \subset \cdots \subset \mathcal{P}_n$ has length $\dim A/\mathcal{P}$, so the thesis follows. □

Note that the first part of Theorem 7.2.4 follows from Corollary 8.3.1, 2. and the second part is Corollary 8.3.2.

If A is any integral domain, the property that all non-extendable chains of prime ideals of A have the same length does not hold in general. There are even examples (not easy to construct) of noetherian domains whose Krull dimension is not finite or where there are non-extendable chains of prime ideals of different lengths. The rings where the property in Corollary 8.3.1 (3) holds are called *catenary* rings.

Chapter 9

Regular and rational functions.

9.1 Regular functions

In this chapter, we will define the regular functions on algebraic varieties, not only on closed subsets of affine or projective space, but more in general on locally closed subsets. This will allow to associate to any algebraic variety an algebraic invariant, its ring of regular functions. An analogous construction will be given also for a more general class of functions, the rational functions, that will bring to a second invariant, the field of rational functions.

Let $X \subset \mathbb{P}^n$ be a locally closed subset and P be a point of X. Let $\varphi : X \to K$ be a function.

Definition 9.1.1. 1. φ is *regular at* P if there exists a suitable neighbourhood of P in which φ can be expressed as a quotient of homogeneous polynomials of the same degree; more precisely, if there exist an open neighbourhood U of P in X and homogeneous polynomials F, $G \in K[x_0, x_1, \dots, x_n]$ with $\deg F = \deg G$, such that $U \cap V_P(G) = \emptyset$ and $\varphi(Q) = F(Q)/G(Q)$, for all $Q \in U$. Note that the quotient $F(Q)/G(Q)$ is well defined.

2. φ is *regular on* X if φ is regular at every point P of X.

This definition of regular function is of *local* character; we can express it saying that φ is regular if it can locally be expressed by quotients of homogeneous polynomials of the same degree.

The set of regular functions on X is denoted by $\mathcal{O}(X)$: it contains K (identified with the set of constant functions), and can be given the structure of a K–algebra, by the definitions:

$$(\varphi+\psi)(P)=\varphi(P)+\psi(P)$$

$$(\varphi\psi)(P)=\varphi(P)\psi(P),$$

for $P\in X$. (Check that $\varphi+\psi$ and $\varphi\psi$ are indeed regular on X.)

Proposition 9.1.2. *Let $\varphi : X\to K$ be a regular function. Let K be identified with $\mathbb{A}^1$ with Zariski topology. Then φ is continuous.*

Proof. It is enough to prove that $\varphi^{-1}(c)$ is closed in X, $\forall\ c\in K$. For all $P\in X$, choose an open neighbourhood U_P and homogeneous polynomials F_P, G_P such that $\varphi|_{U_P}=F_P/G_P$. Then

$$\varphi^{-1}(c)\cap U_P=\{Q\in U_P|F_P(Q)-cG_P(Q)=0\}=U_P\cap V_P(F_P-cG_P)$$

is closed in U_P. The proposition then follows from:

Lemma 9.1.3. *Let T be a topological space, $T=\cup_{i\in I}U_i$ be an open covering of T, $Z\subset T$ be a subset. Then Z is closed if and only if $Z\cap U_i$ is closed in U_i for all i.*

Proof. Assume that $U_i=X\setminus C_i$ and $Z\cap U_i=Z_i\cap U_i$, with C_i and Z_i closed in X.

Claim: $Z=\bigcap_{i\in I}(Z_i\cup C_i)$, hence it is closed.
In fact: if $P\in Z$, then $P\in Z\cap U_i$ for a suitable i. Therefore $P\in Z_i\cap U_i$, so $P\in Z_i\cup C_i$. If $P\notin Z_j\cap U_j$ for some j, then $P\notin U_j$ so $P\in C_j$ and therefore $P\in Z_j\cup C_j$.

Conversely, if $P\in\bigcap_{i\in I}(Z_i\cup C_i)$, then $\forall\ i$, either $P\in Z_i$ or $P\in C_i$. Since $\exists j$ such that $P\in U_j$, hence $P\notin C_j$, so $P\in Z_j$, so $P\in Z_j\cap U_j=Z\cap U_j$. □

□

Corollary 9.1.4. *1. Let $\varphi\in\mathcal{O}(X)$: then $\varphi^{-1}(0)$ is closed. It is denoted $V(\varphi)$ and called the set of zeros of φ.*

2. Let X be a quasi–projective (irreducible) variety and φ, $\psi\in\mathcal{O}(X)$. Assume that there exists U, open non –empty subset such that $\varphi|_U=\psi|_U$. Then $\varphi=\psi$.

Proof. 1. is clear. To prove 2. we note that $\varphi-\psi\in\mathcal{O}(X)$ so $V(\varphi-\psi)$ is closed. By assumption $V(\varphi-\psi)\supset U$, which is dense, because X is irreducible. So $V(\varphi-\psi)=X$.

□

If $X\subset\mathbb{A}^n$ is locally closed in an affine space, we can use on X both homogeneous and non–homogeneous coordinates. If φ is a regular function according to Definition 9.1.1, from a local expression of φ of the form F/G, with F,G homogeneous of the same degree on an open subset of X, we pass to the expression ${}^aF/{}^aG$ for the same function in non-homogeneous coordinates. Note that now ${}^aF, {}^aG$ are no longer homogeneous nor of the same degree, in general.

On the other hand, assume we have a function on X locally represented by quotients of polynomials in n variables; if A/B is such a local expression, with $\deg A = a, \deg = b$, $a \leq b$, the same function is represented in homogeneous coordinates by the following quotient of homogeneous polynomials of the same degree: $(x_0^{a-b})^h A/{}^h B$. Similarly if $a \geq b$.

From this discussion it follows that all polynomial functions are regular: for instance, if $F(x_1, \ldots, x_n)$ is a polynomial of degree d, the polynomial function defined by F can be expressed in homogeneous coordinates in the form $\frac{{}^hF(x_0,\ldots,x_n)}{x_0^d}$. In particular, if X is an affine variety, $K[X] \subset \mathcal{O}(X)$.

If $\alpha \subset K[X]$ is an ideal, we can consider $V(\alpha) := \bigcap_{\varphi\in\alpha} V(\varphi)$: it is closed into X. Note that α is of the form $\alpha = \overline{\alpha}/I(X)$, where $\overline{\alpha}$ is the inverse image of α in the canonical epimorphism, it is an ideal of $K[x_1, \ldots, x_n]$ containing $I(X)$, hence $V(\alpha) = V(\overline{\alpha}) \cap X = V(\overline{\alpha})$.

If K is algebraically closed, the following relative form follows immediately from the Nullstellensatz.

Proposition 9.1.5 (Relative Nullstellensatz). *Let K be an algebraically closed field, let X be an affine variety closed in $\mathbb{A}^n_K$ and $K[X]$ its coordinate ring.*

1. *If $\alpha \subset K[X]$ is a proper ideal then $V(\alpha) \neq \emptyset$.*
2. *If $f \in K[X]$ and f vanishes at all points $P \in X$ such that $g_1(P) = \cdots = g_m(P) = 0$ $(g_1, \ldots, g_m \in K[X])$, then $f^r \in \langle g_1, \ldots, g_m\rangle \subset K[X]$, for some $r \geq 1$.*

Theorem 9.1.6. *Let K be an algebraically closed field. Let $X \subset \mathbb{A}^n_K$ be closed in the Zariski topology. Then $\mathcal{O}(X) \simeq K[X]$. It is an integral domain if and only if X is irreducible.*

Proof. We have already noticed that $K[X] \subset \mathcal{O}(X)$. It remains to prove the opposite inclusion. So let $f \in \mathcal{O}(X)$.

(i) Assume first that X is irreducible. For all $P \in X$ fix an open neighbourhood U_P of P and polynomials F_P, G_P such that $V_P(G_P) \cap U_P = \emptyset$ and $f|_{U_P} = F_P/G_P$. Let f_P, g_P be the functions in $K[X]$ defined by F_P and G_P. Then $g_P f = f_P$ holds on U_P, so it holds on X (by Corollary 9.1.4 (2), because X is irreducible). Let $\alpha \subset K[X]$ be the ideal $\alpha = \langle g_P\rangle_{P\in X}$, generated by all denominators of the various local expressions of φ; α has no zeros on X, because for any P $g_P(P) \neq 0$, so $\alpha = K[X]$. Therefore there exist suitable polynomial functions $h_P \in K[X]$ such that $1 = \sum_{P\in X} h_P g_P$ (sum with finite support). Hence in $\mathcal{O}(X)$ we have the relation: $f = f \sum h_P g_P = \sum h_P(g_P f) = \sum h_P f_P \in K[X]$.

(ii) Let X be reducible: from $g_P f = f_P$ on U_P, we cannot deduce that the same equality holds on X. The idea is to change suitably the local expressions. For any $P \in X$, there exists $R \in K[x_1, \dots, x_n]$ such that $R(P) \neq 0$ and $R \in I(X \setminus U_P)$, so $r \in \mathcal{O}(X)$ is zero outside U_P. So $r g_P f = f_P r$ on X and we conclude as above, after replacing g_P with $g_P r$ and f_P with $f_P r$. □

The characterization of regular functions on projective varieties is completely different: we will see in Theorem 15.2.2 that, if X is an irreducible projective variety, then $\mathcal{O}(X) \simeq K$, i.e. the unique regular functions are constant.

This gives the motivation for introducing the following weaker concept of rational function.

9.2 Rational functions

Definition 9.2.1. Let X be a quasi–projective variety. A *rational function* on X is a germ of regular functions on some open non–empty subset of X.

Precisely, let $\mathcal{K}$ be the set $\{(U, f) |\ U \neq \emptyset, \text{open subset of } X, f \in \mathcal{O}(U)\}$. The following relation on $\mathcal{K}$ is an equivalence relation:

$$(U, f) \sim (U', f') \text{ if and only if } f|_{U\cap U'} = f'|_{U\cap U'}.$$

Reflexive and symmetric properties are quite obvious. Transitive property: let $(U, f) \sim (U', f')$ and $(U', f') \sim (U'', f'')$. Then $f|_{U\cap U'} = f'|_{U\cap U'}$ and $f'|_{U'\cap U''} = f''|_{U'\cap U''}$, hence $f|_{U\cap U'\cap U''} = f''|_{U\cap U'\cap U''}$. $U \cap U' \cap U''$ is a non–empty open subset of $U \cap U''$, which is irreducible and quasi–projective, so by Corollary 9.1.4 $f|_{U\cap U''} = f''|_{U\cap U''}$.

Let $K(X) := \mathcal{K}/\sim$: its elements are by definition the rational functions on X. $K(X)$ can be given the structure of a field in the following natural way.

Let $\langle U, f\rangle$ denote the class of (U, f) in $K(X)$. We define:

$$\langle U, f\rangle + \langle U', f'\rangle = \langle U \cap U', f + f'\rangle,$$

$$\langle U, f\rangle\langle U', f'\rangle = \langle U \cap U', ff'\rangle$$

(check that the definitions are well posed!).

There is a natural inclusion: $K \to K(X)$ such that $c \to \langle X, c\rangle$. Moreover, if $\langle U, f\rangle \neq 0 = \langle X, 0\rangle$, then $U \setminus V(f)$ is not empty, so there exists $\langle U, f\rangle^{-1} = \langle U \setminus V(f), f^{-1}\rangle$: the axioms of a field are all satisfied.

There is also a natural injective map: $\mathcal{O}(X) \to K(X)$ such that $\varphi \to \langle X, \varphi\rangle$.

Summarizing, $K(X)$ is a field, called the **field of rational functions** of the quasi-projective variety X. It is an extension of the base field K, and contains the ring of regular functions $\mathcal{O}(X)$.

Proposition 9.2.2. *If $X \subset \mathbb{A}^n$ is an irreducible affine variety over an algebraically closed field, then $K(X) \simeq Q(\mathcal{O}(X)) = Q(K[X]) = K(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are the coordinate functions on X.*

Proof. The isomorphism is as follows:

(i) $\psi : K(X) \to Q(\mathcal{O}(X))$

If $\langle U, \varphi\rangle \in K(X)$, then there exists $V \subset U$, open and non–empty, such that $\varphi\,|_V = F/G$, where $F, G \in K[x_1, \ldots, x_n]$ and $V(G) \cap V = \emptyset$. We set $\psi(\langle U, \varphi\rangle) = f/g$.

(ii) $\psi' : Q(\mathcal{O}(X)) \to K(X)$

If $f/g \in Q(\mathcal{O}(X))$, we set $\psi'(f/g) = \langle X \setminus V(g), f/g\rangle$.

It is easy to check that ψ and ψ' are well defined and inverse each other. □

Corollary 9.2.3. *If X is an irreducible affine variety over an algebraically closed field, then $\dim X$ is equal to the transcendence degree over K of its field of rational functions.*

Proof. It follows from Corollary 7.2.5. □

Proposition 9.2.4. *If X is a quasi–projective variety and $U \neq \emptyset$ is an open subset, then $K(X) \simeq K(U)$.*

Proof. We have the maps: $K(U) \to K(X)$ such that $\langle V, \varphi\rangle \to \langle V, \varphi\rangle$, and $K(X) \to K(U)$ such that $\langle A, \psi\rangle \to \langle A \cap U, \psi\,|_{A\cap U}\rangle$: they are K–homomorphisms inverse each other. □

Note. The term K-homomorphism means that the elements of K remain fixed.

Corollary 9.2.5. *If X is an irreducible projective variety contained in $\mathbb{P}^n$, if i is an index such that $X \cap U_i \neq \emptyset$ (where U_i is the open subset where $x_i \neq 0$), then $\dim X = \dim X \cap U_i = tr.d.K(X)/K$.*

Proof. By Proposition 7.1.3, $\dim X = \sup_i \dim(X \cap U_i)$. By Corollary 9.2.3 and Proposition 9.2.4, if $X \cap U_i$ is non–empty, $\dim(X \cap U_i) = tr.d.K(X \cap U_i)/K = tr.d.K(X)/K$: it is independent of i. □

If $\langle U, \varphi\rangle \in K(X)$, we can consider all possible representatives of it, i.e. all pairs $\langle U_i, \varphi_i\rangle$ such that $\langle U, \varphi\rangle = \langle U_i, \varphi_i\rangle$. Then $\overline{U} = \bigcup_i U_i$ is the maximum open subset of X on which φ can be seen as a function: it is called the *domain of definition* (or of regularity) of $\langle U, \varphi\rangle$, or simply of φ. It is sometimes denoted domφ. If $P \in \overline{U}$, we say that φ *is regular at* P.

We can consider the set of all rational functions on X which are regular at P: it is denoted by $\mathcal{O}_{P,X}$. It is a subring of $K(X)$ containing $\mathcal{O}(X)$, called the *local ring of* X *at* P. In fact, $\mathcal{O}_{P,X}$ is a local ring, whose maximal ideal, denoted $\mathcal{M}_{P,X}$, is the set of rational functions φ such that $\varphi(P)$ is defined and $\varphi(P) = 0$. To see this, observe that an element of $\mathcal{O}_{P,X}$ can be represented as $\langle U, F/G\rangle$: its inverse in $K(X)$ is $\langle U \setminus V_P(F), G/F\rangle$, which belongs to $\mathcal{O}_{P,X}$ if and only if $F(P) \neq 0$. We will see in Section 9.3 that $\mathcal{O}_{P,X}$ is the localization $K[X]_{I_X(P)}$.

As in Proposition 9.2.4 for the fields of rational functions, also for the local rings of points it can easily be proved that, if $U \neq \emptyset$ is an open subset of X containing P, then $\mathcal{O}_{P,X} \simeq \mathcal{O}_{P,U}$. So the ring $\mathcal{O}_{P,X}$ only depends on the local behaviour of X in the neighbourhood of P.

The *residue field* of $\mathcal{O}_{P,X}$ is the quotient $\mathcal{O}_{P,X}/\mathcal{M}_{P,X}$. This field results to be naturally isomorphic to the base field K. In fact consider the evaluation map $\mathcal{O}_{P,X} \to K$ such that φ goes to $\varphi(P)$: it is surjective with kernel $\mathcal{M}_{P,X}$, so $\mathcal{O}_{P,X}/\mathcal{M}_{P,X} \simeq K$.

Example 9.2.6. 1. The cuspidal cubic.

Let $X \subset \mathbb{A}^2$ be the curve $V(x_1^3 - x_2^2)$. Then $F = x_2$, $G = x_1$ define the function $\varphi = x_2/x_1$ which is regular at the points $P(a_1, a_2)$ such that $a_1 \neq 0$. Another representation of the same function is $\varphi = x_1^2/x_2$, which shows that φ is regular at P if $a_2 \neq 0$. If φ admits another representation F'/G', then $G'x_2 - F'x_1$ vanishes on an open subset of X, which is irreducible (see Exercise 2, Chapter 6), hence $G'x_2 - F'x_1$ vanishes on X, and therefore $G'x_2 - F'x_1 \in \langle x_1^3 - x_2^2\rangle$. We can write $G'x_2 - F'x_1 = H(x_1, x_2)(x_1^3 - x_2^2)$, for a suitable H, so $(G' + Hx_2)x_2 = (F' + Hx_1^2)x_1$. By the UFD property, it follows that there exists $A(x_1, x_2)$ such that $G' + Hx_2 = x_1A$, $F' + Hx_1^2 = x_2A$, so $(F', G') = (x_2A - x_1^2H, x_1A - x_2H) = A(x_2, x_1) - H(x_1^2, x_2)$.

This shows that there are essentially only the above two representations of φ. So $\varphi \in K(X)$ and its domain of regularity is $X \setminus \{0, 0\}$. We will see later (Example 10.1.2) another way to explain why the domain of definition cannot be all X.

2. The stereographic projection.

Let $X \subset \mathbb{P}^2$ be the curve $V_P(x_1^2 + x_2^2 - x_0^2)$. Let $f := x_1/(x_0 - x_2)$ denote the germ of the regular function defined by $x_1/(x_0 - x_2)$ on $X \setminus V_P(x_0 - x_2) = X \setminus \{[1, 0, 1]\} = X \setminus \{P\}$. On X we have $x_1^2 = (x_0 - x_2)(x_0 + x_2)$ so f is represented also as $(x_0 + x_2)/x_1$ on

$X \setminus V_P(x_1) = X \setminus \{P, Q\}$, where $Q = [1, 0, -1]$. If we identify K with the affine line $V_P(x_2) \setminus V_P(x_0)$ (the points of the x_1–axis lying in the affine plane U_0), then f can be interpreted as the stereographic projection of X centered at P, which takes $A[a_0, a_1, a_2]$ to the intersection of the line AP with the line $V_P(x_2)$. To see this, observe that AP has equation $a_1x_0 + (a_2 - a_0)x_1 - a_1x_2 = 0$; and $AP \cap V_P(x_2)$ is the point $[a_0 - a_2, a_1, 0]$.

9.3 Algebraic characterization of the local ring $\mathcal{O}_{P,X}$.

Let us recall the construction of the *ring of fractions of a ring* A with respect to a multiplicative subset S.

Let A be a ring and $S \subset A$ be a multiplicative subset. The following relation in $A \times S$ is an equivalence relation:

$$(a, s) \simeq (b, t) \text{ if and only if } \exists\, u \in S \text{ such that } u(at - bs) = 0.$$

Then the quotient $A \times S/_{\simeq}$ is denoted $S^{-1}A$ or A_S and $[(a, s)]$ is denoted $\frac{a}{s}$. A_S becomes a commutative ring with unit with operations $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$ (check that they are well–defined). With these operations, A_S is called the ring of fractions of A with respect to S, or the *localization* of A in S.

There is a natural homomorphism $j : A \to S^{-1}A$ such that $j(a) = \frac{a}{1}$, which makes $S^{-1}A$ an A–algebra (in the sense that it contains a homomorphic image of A). Note that j is the zero map if and only if $0 \in S$. More precisely if $0 \in S$ then $S^{-1}A$ is the zero ring: this case will always be excluded in what follows. Moreover j is injective if and only if every element in S is not a zero divisor. In this case $j(A)$ will be identified with A.

Example 9.3.1.

1. Let A be an integral domain and set $S = A \setminus \{0\}$. Then $A_S = Q(A)$: the quotient field of A.

2. If $\mathcal{P} \subset A$ is a prime ideal, then $S = A \setminus \mathcal{P}$ is a multiplicative set and A_S is denoted $A_{\mathcal{P}}$ and called the localization of A at $\mathcal{P}$.

3. If $f \in A$, then the multiplicative set generated by f is

$$S = \{1, f, f^2, \dots, f^n, \dots\} :$$

A_S is denoted A_f.

4. If $S = \{x \in A \mid x \text{ is regular}\}$, then A_S is called the total ring of fractions of A: it is the maximum ring in which A can be canonically embedded.

It is easy to verify that the ring A_S enjoys the following *universal property*:

(i) if $s \in S$, then $j(s)$ is invertible;

(ii) if B is a ring with a given homomorphism $f : A \to B$ such that for any $s \in S$ $f(s)$ is invertible, then f factorizes through A_S, i.e. there exists a unique homomorphism $\overline{f}$ such that $\overline{f} \circ j = f$.

We will see now the relations between ideals of A_S and ideals of A.

If $\alpha \subset A$ is any ideal, then $\alpha A_S = \{\frac{a}{s} \mid a \in \alpha\}$ is called the *extension of* α in A_S and denoted also α^e. It is an ideal, precisely the ideal generated by the set $\{\frac{a}{1} \mid a \in \alpha\}$.

If $\beta \subset A_S$ is an ideal, then $j^{-1}(\beta) =: \beta^c$ is called the contraction of β and is clearly an ideal.

The following Proposition gives the complete picture.

Proposition 9.3.2. *1. For any ideal $\alpha \subset A : \alpha^{ec} \supset \alpha$;*

2. for any ideal $\beta \subset A_S : \beta = \beta^{ce}$;

3. α^e is proper if and only if $\alpha \cap S = \emptyset$;

4. $\alpha^{ec} = \{x \in A \mid \exists s \in S$ such that $sx \in \alpha\}$.

Proof. 1. and 2. are straightforward.

3. if $1 = \frac{a}{s} \in \alpha^e$, then there exists $u \in S$ such that $u(s - a) = 0$, i.e. $us = ua \in S \cap \alpha$. Conversely, if $s \in S \cap \alpha$ then $1 = \frac{s}{s} \in \alpha^e$.

4.

$$\alpha^{ec} = \{x \in A \mid j(x) = \frac{x}{1} \in \alpha^e\} =$$

$$= \{x \in A \mid \exists a \in \alpha, t \in S \text{ such that } \frac{x}{1} = \frac{a}{t}\} =$$

$$= \{x \in A \mid \exists a \in \alpha, t, u \in S \text{ such that } u(xt - a) = 0\}.$$

Hence, if $x \in \alpha^{ec}$, then: $(ut)x = ua \in \alpha$. Conversely: if there exists $s \in S$ such that $sx = a \in \alpha$, then $\frac{x}{1} = \frac{a}{s}$, i.e. $j(x) \in \alpha^e$. □

If α is an ideal of A such that $\alpha = \alpha^{ec}$, α is called *saturated* with S. For example, if $\mathcal{P}$ is a prime ideal and $S \cap \mathcal{P} = \emptyset$, then $\mathcal{P}$ is saturated and $\mathcal{P}^e$ is prime. Conversely, if $\mathcal{Q} \subset A_S$ is a prime ideal, then $\mathcal{Q}^c$ is prime in A.

Corollary 9.3.3. *There is a bijection between the set of prime ideals of A_S and the set of prime ideals of A not intersecting S. In particular, if $S = A \setminus \mathcal{P}$, $\mathcal{P}$ prime, the prime ideals of $A_\mathcal{P}$ correspond bijectively to the prime ideals of A contained in $\mathcal{P}$, hence $A_\mathcal{P}$ is a local ring with maximal ideal $\mathcal{P}^e$, denoted $\mathcal{P}A_\mathcal{P}$, and residue field $A_\mathcal{P}/\mathcal{P}A_\mathcal{P}$. Moreover* $\dim A_\mathcal{P} = \mathrm{ht}\mathcal{P}$.

In particular we get the characterization of $\mathcal{O}_{P,X}$. Let $X \subset \mathbb{A}^n$ be an affine variety, let P be a point of X and $I(P) \subset K[x_1, \dots, x_n]$ be the ideal of P. Let $I_X(P) := I(P)/I(X)$ be the ideal of $K[X]$ formed by the regular functions on X vanishing at P. Then we can construct the localization

$$\mathcal{O}(X)_{I_X(P)} = \left\{ \frac{f}{g} \mid f, g \in \mathcal{O}(X), g(P) \neq 0 \right\} \subset K(X).$$

It is canonically identified with $\mathcal{O}_{P,X}$. In particular, if K is algebraically closed:

$$\dim \mathcal{O}_{P,X} = \text{ht } I_X(P) = \dim \mathcal{O}(X) = \dim X.$$

There is a bijection between prime ideals of $\mathcal{O}_{P,X}$ and prime ideals of $\mathcal{O}(X)$ contained in $I_X(P)$; they also correspond to prime ideals of $K[x_1, \dots, x_n]$ contained in $I(P)$ and containing $I(X)$.

If X is an affine variety, it is possible to define the local ring $\mathcal{O}_{P,X}$ also if X is reducible, in a purely algebraic way, simply as localization of $K[X]$ at the maximal ideal $I_X(P)$. The natural map j from $K[X]$ to $\mathcal{O}_{P,X}$ is injective if and only if $K[X] \setminus I_X(P)$ does not contain any zero divisor. A non-zero function f is a zero divisor in $K[X]$ if there exists a non-zero g such that $fg = 0$, i.e. $X = V(f) \cup V(g)$ is an expression of X as union of proper closed subsets. For j to be injective it is required that every zero divisor f belongs to $I_X(P)$, which means that all the irreducible components of X pass through P.

Exercises 9.3.4. 1. Prove that the irreducible affine varieties and the open subsets of irreducible affine varieties are quasi–projective varieties.

2. Let $X = \{P, Q\}$ be the union of two points in an affine space over K. Prove that $\mathcal{O}(X)$ is isomorphic to $K \times K$.