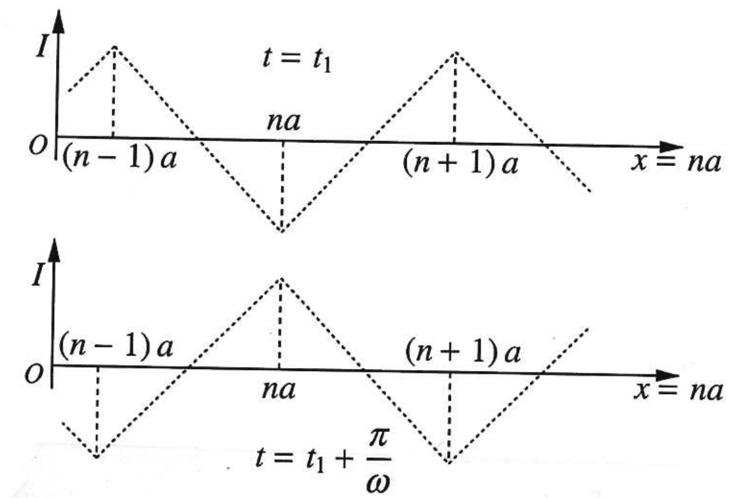
FREQUENCY $\omega_{MAX} = 2\omega_0 \Rightarrow \omega > \omega_{MAX}$ CANNOT BE TRANSMITTED $\Rightarrow$ A LC NETWORK ACTS AS A LOW PASS FILTER.



A GENERAL MONOCHROMATIC SOLUTION

THE GENERAL MONOCHROMATIC SOLUTION OF FREQUENCY ω IS A STANDING WAVE, I.E. THE SUM OF TWO WAVES. ONE PROGRESSIVE AND THE OTHER REGRESSIVE

$$I_n(t) = A e^{i(kna-\omega t)} + B e^{-i(kna-\omega t)} \quad (230)$$

WHERE ω AND k ARE RELATED BY THE DISPERSION RELATION. BECAUSE OF OUR BOUNDARY CONDITIONS WE MUST HAVE

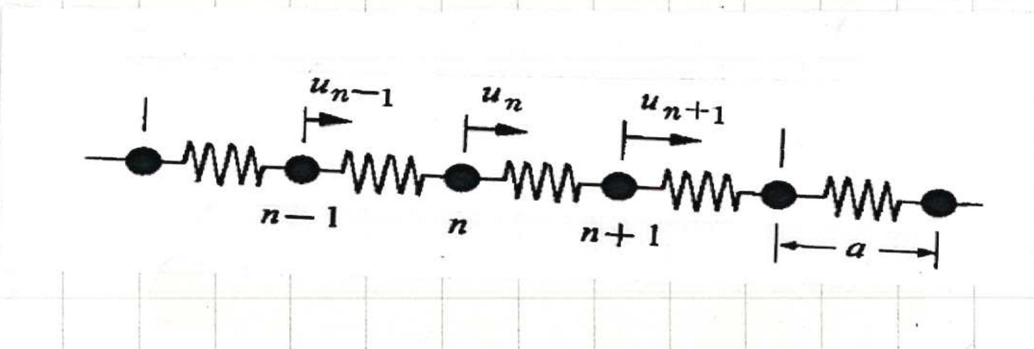
$X_0(t) = 0 \Rightarrow A + B = 0$; $X_N(t) = 0 \Rightarrow$

$A e^{ikNa} + B e^{-ikNa} = 0$. THIS GIVES THE CONDITION $e^{ikNa} - e^{-ikNa} = 2i \sin(kNa) = 0$

i.e., $K = \pi \ell / Na$ WITH $\ell = 1, 2, 3, \dots, N-1, N$.
WE HAVE N ALLOWED WAVEVECTR K_ℓ AND FRE-
QUENCY $\omega_\ell = \omega(K_\ell)$. NOTE THAT $K_{min} = \pi / Na$
CORRESPONDS TO $\lambda_{MAX} = 2\pi / K_{min} = 2Na$.
THIS IS A STANDING WAVE OF WAVELENGTH
TWICE THE LENGTH OF THE SYSTEM.

- CLASSICAL THEORY OF A CRYSTAL LATTICE VIBRATIONS

THE MAIN INFORMATION WE GAIN FROM THE LC LADDER NETWORK IS THE REPRESENTATION OF THE PHYSICAL PROPERTIES OF AN INFINITE TRANSLATIONAL SYMMETRY SYSTEM WITHIN THE RECIPROCAL LATTICE UNIT CELL (IN SOLIDS IS KNOWN AS THE BRILLOUIN ZONE). LET'S START WITH A 1D CHAIN OF ATOMS (IONS) CONNECTED WITH EQUAL SPRINGS, AS SHOWN IN THE FIGURE. IF M IS THE MASS AND WE CONSIDER ONLY THE LONGITUDINAL MOTION



THE MOTION EQUATION $m\bar{a} = F = m\ddot{x} = KX$

$$M\ddot{u}_n = \alpha(u_{n+1} - u_n) - \alpha(u_n - u_{n-1}) \qquad (231)$$

WHERE u_n IS THE COORDINATE OF THE n-th

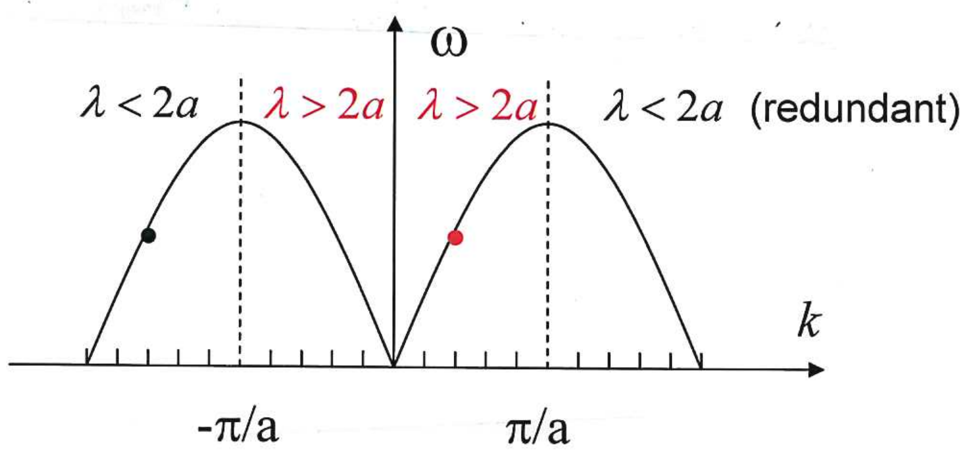
ATOM AND α IS THE SPRING CONSTANT.
SOLUTION OF THE MOTION EQUATION IS

(232) $u_n = A \exp i(kx_n - \omega t)$ WHERE $x_n = na$,
$\omega = \sqrt{k/M}$. THE ASSOCIATED ALGEBRIC EQ. IS

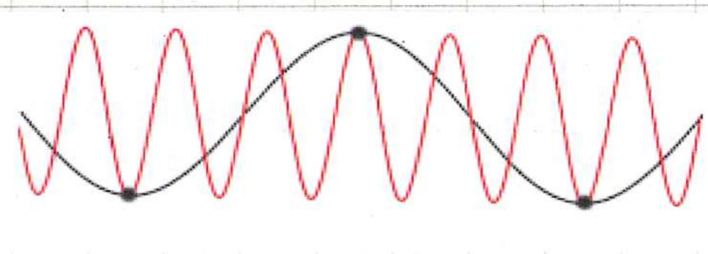
$$M(-\omega^2)e^{ikna} = -\alpha\left[2e^{ikna} - e^{ik(n+1)a} - e^{ik(n-1)a}\right] \quad (233)$$

WHICH LEADS TO $\omega(k) = \omega_M |\sin(ka/2)|$, WITH

$\omega_M = 2\sqrt{\alpha/M}$. IN ANALOGY TO WHAT DONE FOR THE LC LADDER NETWORK WE CAN DRAW THE FUNCTION THAT REPRESENTS THE DISPERSION CURVE $\boxed{\omega(k) = \omega_M |\sin(ka/2)|}$ (234)



THE WAVE WITH WAVE NUMBERS k AND $k + \frac{2\pi}{a}$ DESCRIBE THE SAME ATOMIC DISPLACEMENT.



THEREFORE WE CAN RESTRICT K TO THE FIRST BRILLOUIN ZONE (BZ) $\left[-\pi/a, \pi/a\right]$

- OBSERVATION THE DISPLACEMENT OF THE n-ATOM

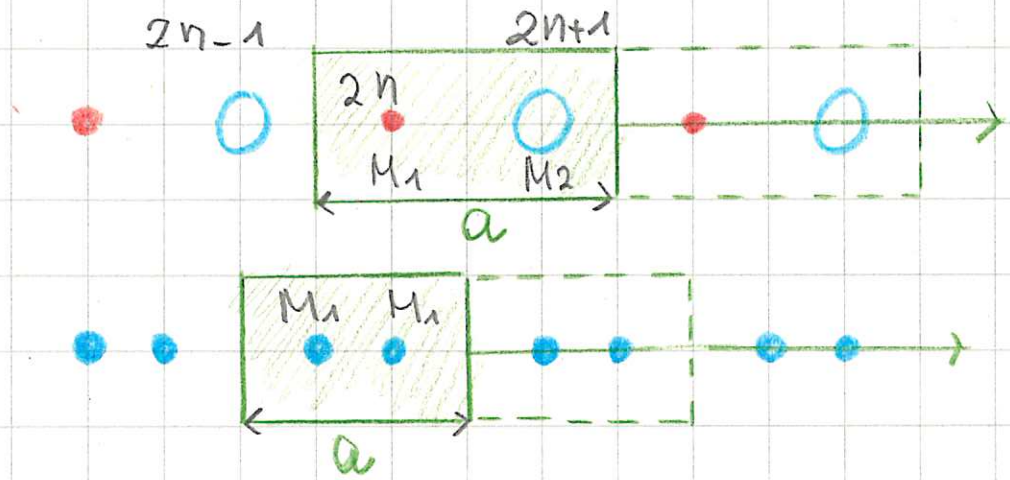
$$u_n(t) = A \exp i(kX_n - \omega t), \; X_n = na$$

- PATTERN OF VIBRATION
  - $K \simeq 0 \Rightarrow \exp(ikX_n) \sim 1 \Rightarrow$ EVERY ATOMS MOVE AT UNISON (IN PHASE) $\Rightarrow$ LITTLE RESTORING FORCE
  - $K \simeq \pi/a$, $\exp(ikX_n) \simeq (-1)^n \Rightarrow$ ADIACENT ATOMS MOVE IN OPPOSITE DIRECTIONS $\Rightarrow$ MAXIMUM RESTORING FORCE.
- VELOCITY OF WAVES
  - $K \simeq 0 \Rightarrow \omega = (\omega_M a/2) k \Rightarrow$ LINEAR DISPERSION $\Rightarrow$ PHASE VELOCITY = GROUP VELOCITY
  - $K \simeq \pi/a \Rightarrow$ GROUP VELOCITY = 0
- NUMBER OF "NORMAL MODES"

$$u_N = u_0 \Rightarrow \exp(ikNa) = 1 \quad \Delta K = 2\pi/Na$$

- VIBRATION WITH TWO ATOMS PER UNIT CELL

WE MUST MAKE CLEAR THAT TWO ATOMS PER UNIT CELL CAN BE INTERPRETED AS TWO ATOMS WITH DIFFERENT MASS (M_1, M_2 LIKE IN GaAs) OR TWO ATOMS WITH THE SAME MASS BUT DIFFERENT COORDINATES.



WE WILL TREAT THE CASE OF TWO DIFFERENT MASSES FIRST (M_1, M_2). FOR THIS CASE WE HAVE TWO MOTION EQS.

$$M_2 \ddot{u}_{2n+1} = -\alpha\left(2u_{2n+1} - u_{2n} - u_{2n+2}\right)$$

$$M_1 \ddot{u}_{2n+2} = -\alpha\left(2u_{2n+2} - u_{2n+1} - u_{2n+3}\right) \qquad (235)$$

WE ASSUMA AS SOLUTIONS $\begin{bmatrix} u_{2n+1} \\ u_{2n+2} \end{bmatrix} = \begin{bmatrix} A_1 e^{-ik\bar{X}_{2n+1}} \\ A_2 e^{ik\bar{X}_{2n+2}} \end{bmatrix} e^{-i\omega t}$

$$\bar{X}_{2n+1} = (2n+1)\,a/2$$

(236)

$$\bar{X}_{2n+2} = (n+1)\,a$$

$$\Rightarrow D \begin{bmatrix} 2\alpha - M_2\omega^2 & -2\alpha\cos\left(\frac{ka}{2}\right) \\ -2\alpha\cos\left(\frac{ka}{2}\right) & 2\alpha - M_1\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

DET.

$$\Rightarrow \omega_{\pm}^2 = \alpha\left(\frac{1}{M_1} + \frac{1}{M_2}\right) \pm \alpha\sqrt{\left(\frac{1}{M_1} + \frac{1}{M_2}\right)^2 - \frac{4\sin^2(ka/2)}{M_1 M_2}} \qquad (237)$$

THIS IMPLIES TWO DISPERSION BRACHES

WITH $M_1 = M_2$ WE OBTAIN

$$\omega_{\pm}^2 = \alpha\left(\frac{2}{M}\right) \pm \alpha\sqrt{\frac{4}{M^2} - \frac{4\sin^2(ka/2)}{M^2}} \qquad (238)$$

$$\omega_{\pm}^2 = \alpha\left(\frac{2}{M}\right) \pm \alpha 2M\sqrt{1-\sin^2(ka/2)}$$ (239) 131

$$= \alpha\left(\frac{2}{M}\right) \pm \alpha 2M\cos\left(\frac{ka}{2}\right)$$

$$k=0 \Rightarrow \omega_{\pm}^2 = \alpha\left(\frac{2}{M}\right) \pm \alpha\, 2/M \nearrow 0 \quad \searrow 4\alpha/M \Rightarrow \omega = 2\sqrt{\frac{\alpha}{M}}$$

$$k=\pm\pi/a \Rightarrow \omega_{\pm}^2 = 2\alpha/M \Rightarrow \omega = \sqrt{\frac{2\alpha}{M}}$$

• OBSERVATION IF $M_1 \neq M_2$ AT THE BOARDER OF THE ZONE ($k = \pm\pi/a$) $\omega_+^2 > \omega_-^2$ THUS WE HAVE A GAP. WHEREAS AT $k=0$ WE HAVE ONLY ONE REAL ω (ω_+).

IF $M_1 \neq M_2$ AT THE BOARDER ZONE $\omega_+(\pm\pi/2) = \omega_-(\pm\pi/2)$, WHEREAS AT $k=0$ WE HAVE ONLY ONE REAL $\omega = 2\sqrt{\frac{\alpha}{M}}$