X quasi-projective variety

$K(X) = \{ \langle U, \varphi \rangle \mid U \subseteq X \text{ open}, \ \varphi \in \mathcal{O}(U) \}$

$\operatorname{dom} \varphi$ $\qquad \mathcal{O}_{X,P} = \{ \varphi \in K(X) \mid P \in \operatorname{dom} \varphi \}$ local ring

Residue field: $\mathcal{O}_{X,P} / m_{X,P} = \kappa(\mathcal{O}_{X,P})$

$m_{X,P} = \{ \varphi \in \mathcal{O}_{X,P} \mid \varphi(P) = 0 \}$

$$\begin{aligned} ev : \mathcal{O}_{X,P} &\longrightarrow K \\ \varphi &\longmapsto \varphi(P) \end{aligned}$$

evaluation map, surjective

K-homomorphism
$\shortparallel$
ring homom. s.t. fixes the elem. of K

$\ker(ev) = m_{X,P}$

$\Rightarrow K \simeq \mathcal{O}_{X,P} / m_{X,P} = \kappa(\mathcal{O}_{X,P})$

Example $X \subseteq \mathbb{A}^2$ cuspidal cubic $x_1^3 - x_2^2 = 0$

$\varphi = \frac{x_2}{x_1}$ function on $U = X - V(x_1) = X - \{(0,0)\}$

$= \frac{x_1^2}{x_2}$ on U $\quad x_1^3 = x_2^2 \Rightarrow \frac{x_2}{x_1} = \frac{x_1^2}{x_2}$

Can we find other expressions for φ?

Ans. $V \subseteq X$ open $\quad \varphi = \frac{F}{G}$ on $V \Rightarrow$ on $U \cap V$ $\frac{F}{G} = \frac{x_2}{x_1}$

$\Rightarrow \underline{Fx_1 - Gx_2 = 0}$ on $U \cap V \subseteq X$ irred.

$\Rightarrow Fx_1 - Gx_2 = 0$ on X

$\Rightarrow Fx_1 - Gx_2 \in I(X) = (x_1^3 - x_2^2)$

$Fx_1 - Gx_2 = H(x_1^3 - x_2^2)$, $\quad H \in k[x_1, x_2]$

$Fx_1 - Hx_1^3 = Gx_2 - Hx_2^2$

$\underline{x_1}(F - Hx_1^2) = \underline{x_2}(G - Hx_2) \Rightarrow x_1 \mid G - Hx_2$

$x_2 \mid F - Hx_1^2$

$G - Hx_2 = x_1 A \qquad x_1 x_2 B = x_2 x_1 A$

$F - Hx_1^2 = x_2 B \qquad \Downarrow$

$A = B$

$G = x_1 A + x_2 H$

$F = x_2 A + x_1^2 H \qquad \mathrm{dom}\,\varphi = U$

$(F, G) = A(x_1, x_2) + H(x_2, x_1^2)$

$\frac{F}{G} = \frac{x_2 A + x_1^2 H}{x_1 A + x_2 H} \longrightarrow$ vanishes on $(0,0)$

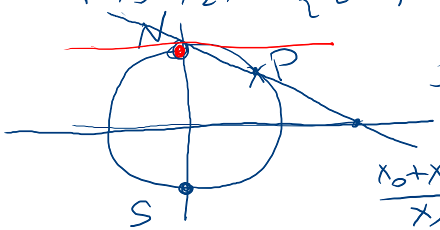
2. Stereographic projection.

$X \subseteq \mathbb{P}^2$ circle $\qquad X: x_1^2+x_2^2-x_0^2=0 \qquad x^2+y^2-1=0$

$f: X \longrightarrow K \qquad f=\dfrac{x_1}{x_0-x_2} \in K(X)$, dom $f \supseteq X - V_p(x_0-x_2)$

$\longrightarrow \quad x_1^2 = x_0^2-x_2^2 = (x_0-x_2)(x_0+x_2)$

$V_p(x_0-x_2) = \{[1,0,1]\} = \{N\}$ North pole



$f=\dfrac{x_1}{x_0-x_2} = \dfrac{x_0+x_2}{x_1}$ as rational function

$\dfrac{x_0+x_2}{x_1}$ is regular where $x_1\neq 0$: $X-\{S,N\}$

$x_1=0 \Longrightarrow x_0^2-x_1^2=0 \Longrightarrow N,\ S=\{[1,0,-1]\}$

dom $f \supseteq X - V_p(x_0-x_2)$

Geometric interpretation: $\qquad K \longleftrightarrow V_p(x_2)\setminus\{[0,1,0]\}$

$\qquad c \longleftrightarrow [1, c, 0]$

$f: P \longmapsto NP \cap V_p(x_2)$, $P \neq N$

$P[a_0,a_1,a_2] \qquad NP: \quad N = [1,0,1]$

$$\begin{cases} x_0 = \lambda a_0 + \mu \\ x_1 = \lambda a_1 \\ x_2 = \lambda a_2 + \mu = 0 \end{cases} \qquad \mu = -\lambda a_2$$

$$\begin{cases} x_0 = \lambda a_0 - \lambda a_2 = \lambda(a_0-a_2) \\ x_1 = \lambda a_1 \\ x_2 = 0 \end{cases} \qquad P \longrightarrow [a_0-a_2, a_1, 0]$$

$$\updownarrow$$

$$\frac{a_1}{a_0-a_2} = f$$

$X \subseteq \mathbb{A}^n$ closed, irred., $P \in X$, $K[X] \supseteq \underset{\text{prime}}{I_X(P)} = I(P) / I(X)$

$K[X]_{I_X(P)} = \mathcal{O}_{X,P}$

$j : K[X] \longrightarrow \mathcal{O}_{X,P}$ injective

$f \longrightarrow \frac{f}{1}$

$\mathfrak{P} \subseteq K[X] \rightsquigarrow \mathfrak{P}^e = \langle \frac{p}{1} \mid p \in \mathfrak{P} \rangle$ extended ideal, prime ideal

$Q \subseteq \mathcal{O}_{X,P}$ $\quad Q^c = Q \cap K[X]$ prime

Bijection: prime ideals of $K[X]$ contained in $I_X(P)$ and prime ideals of $\mathcal{O}_{X,P}$

$\dim \mathcal{O}_{X,P}$: chains of prime ideals in $\mathcal{O}_{X,P}$

$\updownarrow$

chains of prime ideals of $\underline{K[X]}$ contained in $\underline{I_X(P)}$ maximal

$\underline{\dim \mathcal{O}_{X,P}} = \operatorname{ht} I_X(P) = \underline{\dim K[X]}$

$\downarrow$

all maximal ideals have the same length

K alg closed: $\boxed{\dim \mathcal{O}_{X,P} = \dim X}$

$$X = \bigcup_{i=0}^{n} (X \cap U_i)$$

$\boxed{\text{prog. var.}}$

$$\dim X = \underbrace{\dim X \cap U_i}_{\text{if } \neq \emptyset}$$

$$\mathcal{O}_{X,P} \cong \mathcal{O}_{X \cap U_i, P}$$

$$\dim X = \dim X \cap U_i = \dim \mathcal{O}_{X \cap U_i, P} = \underline{\underline{\dim \mathcal{O}_{X,P}}}$$

$\mathcal{O}(X)$

Regular maps or morphisms

X, Y quasi-projective varieties $/K$ not nec. irreducible

$\varphi : X \longrightarrow Y$ a map

<u>Def</u>: φ is regular or is a morphism if:

1) φ is continuous

2) "φ preserves regular functions":

$$X \xrightarrow{\varphi} Y$$

$$\varphi^{-1}(U) \xrightarrow{\varphi} \underset{\cup\!\mathsf{I}}{U} \xrightarrow{f} K$$

$\forall\, U \subseteq Y$ open $\neq \emptyset$,
$\forall\, f \in \mathcal{O}(U)$ we ask that
$f \circ \varphi| \in \mathcal{O}(\varphi^{-1}(U))$

<u>Rmks</u>

→ 1) $1_X : X \longrightarrow X$ is regular

→ 2) If $X \xrightarrow{\varphi} Y$, $Y \xrightarrow{\psi} Z$ are regular maps $\Longrightarrow$

$\psi \circ \varphi$ is regular

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

$$(\psi \circ \varphi)^{-1}(U) = \varphi^{-1}(\psi^{-1}(U))$$

$$\underbrace{\varphi^{-1}(\psi^{-1}(U)) \longrightarrow \psi^{-1}(U)}_{\text{regular}} \underbrace{\longrightarrow \underset{\cup\!\mathsf{I}}{U} \xrightarrow{f}}_{\text{regular}} K$$

$f \circ (\psi \circ \varphi)|$ is reg.

Def. $X \xrightarrow{\varphi} Y$ is an isomorphism if φ is regular and $\exists\ \varphi^{-1}: Y \longrightarrow X$ regular

If $\exists$ isomorphism $\varphi: X \to Y$: X, Y are isomorphic

$X \simeq Y$

Def. $\varphi: X \to Y$ regular $\quad \mathcal{O}(X), \mathcal{O}(Y)$

$\varphi^*: \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ comorphism of φ

$f \longmapsto f \circ \varphi$ it is a K- homomorphism

$X \xrightarrow{\varphi} Y \xrightarrow{f} K$

$\overset{\shortparallel}{\varphi^{-1}(Y)}$

$\varphi^*(f+g) = (f+g)\circ\varphi = f\circ\varphi + g\circ\varphi$

$= \varphi^*(f) + \varphi^*(g)$

$\varphi^*(fg) = \varphi^*(f)\,\varphi^*(g)$

$c \in K \quad \varphi^*(c) = c$

FUNCTORIALITY

2 properties:

1) $X \xrightarrow{1_X} X \qquad 1_X^*: \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$

$1_X^* = 1_{\mathcal{O}(X)} \qquad f \longmapsto f \circ 1_X = f$

2) $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ regular $\qquad \mathcal{O}(Z) \xrightarrow{\psi^*} \mathcal{O}(Y) \xrightarrow{\varphi^*} \mathcal{O}(X)$, $\quad (\psi\circ\varphi)^*$

$\boxed{\varphi^* \circ \psi^* = (\psi\circ\varphi)^*}$

The construction of the comorphism is functorial

Consequence If $\varphi: X \xrightarrow{\sim} Y \Longrightarrow \varphi^*: \mathcal{O}(Y) \to \mathcal{O}(X)$ is a ring isomorphism.

$\psi: Y \to X$ regular $\qquad X \xrightarrow{\varphi} Y \xrightarrow{\psi} X \qquad Y \xrightarrow{\psi} X \xrightarrow{\varphi} Y$

$\boxed{\begin{array}{l} \psi\circ\varphi = 1_X \\ \varphi\circ\psi = 1_Y \end{array}}$

$(\psi\circ\varphi)^* = 1_X^* = 1_{\mathcal{O}(X)} \qquad (\varphi\circ\psi)^* = 1_Y^* = 1_{\mathcal{O}(Y)}$

$\overset{\shortparallel}{\varphi^*\circ\psi^*} \qquad \overset{\shortparallel}{\psi^*\circ\varphi^*}$

φ^*, ψ^* are each one inverse $\to \mathcal{O}(Y) \simeq \mathcal{O}(X)$

$X = V(x^3 - y^2) \qquad K[X] \not\simeq K[t] \to$ variable

$\Downarrow$

$X \not\simeq \mathbb{A}^1$

$X, \mathbb{A}^1$ are homeom. $\qquad \mathbb{A}^1 \longrightarrow X$ bijection

$t \longmapsto (t^2, t^3)$

but $X \not\simeq \mathbb{A}^1$