$\varphi : X \longrightarrow Y$, X, Y locally closed in $\mathbb{P}^n_K, \mathbb{P}^m_K$

φ morphism or regular map if

1) continuous

2) $\forall U \subseteq Y$ open, $f \in \mathcal{O}(U)$

$$\boxed{f \circ \varphi| \in \varphi^{-1}(U)}$$

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Y \\ \varphi^{-1}(U) \arrow[r, "\varphi|_{\varphi^{-1}(U)}"'] & U \arrow[u, phantom, "\cup"] \arrow[r, "f"] & K \end{tikzcd}$$

If $\varphi(X) \subseteq Y \setminus U$

$\varphi^{-1}(U) = \emptyset$

$\varphi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ comorphism : K-homom.

$f \longrightarrow \varphi^*(f) = f \circ \varphi$

Examples

1) X cuspidal cubic in $\mathbb{A}^2_K$ $\mathcal{O}(X) \not\simeq K[t]$ polym.

$\Longrightarrow X \not\simeq \mathbb{A}^1$

$\varphi : \mathbb{A}^1 \longrightarrow X$ φ bijective, continuous, φ^{-1} contin.

$t \longrightarrow (t^2, t^3)$ homeom.

$\varphi^{-1} : X \longrightarrow \mathbb{A}^1$

$(x, y) \longrightarrow \begin{cases} \dfrac{y}{x} & \text{if } (x,y) \neq (0,0) \\ \boxed{0} & \text{if } (x,y) = 0 \end{cases}$ φ^{-1} cannot be regular

$x^3 = y^2$

for φ^{-1} being regular, we should find an expension by polynom. in a nbhd of $(0,0)$

φ is regular

$$\mathbb{A}^1 \longrightarrow X \qquad f\big|_{U_P} = \frac{F}{G} \qquad F, G \in K[x,y]$$

$$t \longmapsto (t^2, t^3)$$

$$\begin{array}{c} \cup\!\mid \\ U \xrightarrow{f} K \end{array} \qquad P \in U$$

$$f \circ \varphi \big|_{\varphi^{-1}(U_P)} = \frac{F(t^2,t^3)}{G(t^2,t^3)}$$ quotient of pol in t $\Longrightarrow$ regular

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$f \circ \varphi\big|_{\varphi^{-1}(U)}$ regular because $\forall P \in U$ there is a nbhd where it is quot. of pol.

φ^{-1} is not regular

2) $\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$, $\forall i$ U_i is homeom. to $\mathbb{A}^n$: it is isomorphic

$i=0$ $\qquad j_0 : \mathbb{A}^n \longrightarrow U_0 \qquad (a_1, \dots, a_n) \longrightarrow [1, a_1 \dots a_n]$

$$j_0^{-1}(U) \xrightarrow{j_0} U \xrightarrow{f} K \qquad f\big|_{U_P} = \frac{F(x_0 \dots x_n)}{G(x_0 \dots x_n)}$$

($U \subseteq U_0$, $P \in U$)

$$f \circ j_0^{-1}(U_P) = \frac{F(1, x_1 \dots)}{G(1, x_1 \dots x_n)} = \frac{aF}{aG}$$

$$(x_1 \dots x_n) \longrightarrow [1, x_1 \dots x_n] \xrightarrow{f} \frac{aF}{aG} \quad \text{reg}$$

$$\varphi_0 : U_0 \longrightarrow \mathbb{A}^n \qquad [x_0 \dots x_n] \longrightarrow \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

($x_0 \neq 0$)

$$\varphi_0^{-1}(U) \xrightarrow{\varphi_0|} U \xrightarrow{f} K \qquad f\big|_{U_P} = \frac{A(x_1 \dots x_n)}{B(x_1 \dots x_n)}$$

$$\varphi_0^{-1}(U_P) \qquad U \supseteq U_P$$

$$[x_0 \dots x_n] \longrightarrow \left(\frac{x_1}{x_0}, \dots\right) \longrightarrow \frac{A\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)}{B\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)} = \frac{x_0^d \, {}^h A}{{}^h B}$$

$\deg A \leq \deg B \qquad d = \deg A - \deg B$

Expression of regular maps in the affine case

$\varphi : X \longrightarrow Y \subseteq A^n$ We can consider $t_1, -, t_n$ coordinate functions on Y, $t_1, .., t_n \in \mathcal{O}(Y)$

$$\varphi^*(t_i) = t_i \circ \varphi : X \longrightarrow K$$
$$P \longrightarrow \varphi(P) \longrightarrow t_i(\varphi(P))$$

$$\varphi = (\varphi^*(t_1), -, \varphi^*(t_n)) = \underline{(\varphi_1, -, \varphi_n)}$$

$$\varphi(P) = (t_1(\varphi(P)), -, t_n(\varphi(P)))$$

φ regular $\Longrightarrow \varphi_1, -, \varphi_n$ are regular on X

Prop. $\varphi : X \to Y \subseteq A^n$ is regular $\Longleftrightarrow \forall i = 1, .., n$
$\varphi^*(t_i) \in \mathcal{O}(X)$, $t_1 .. t_n$ coord. funs. on Y

Pf. Ass. $\varphi^*(t_i)$ regular $\forall i$.

1) φ is continuous, $Z \subseteq Y$ closed $Z = Y \cap V(F_1, -, F_r)$

$$\varphi^{-1}(Z) = \{P \in X \mid F_j(\varphi(P)) = F_j(\varphi^*(t_1)(P), -, \varphi^*(t_n)(P)) = 0 \quad \forall j = 1, -, r$$

$$F_j(\varphi^*(t_1)(P), -, \varphi^*(t_n)(P)) =$$

$$= \underbrace{F_j(\underbrace{\varphi^*(t_1), -, \varphi^*(t_n)}_{\text{regular by assump.}})}_{\in \mathcal{O}(X)}(P) = 0$$

$$\varphi^{-1}(Z) = V(F_1(\varphi^*(t_1), -, \varphi^*(t_n), - - . F_r(\varphi^*(t_1), -, \varphi^*(t_n))$$
closed in X

$\varphi^*(t_1), .., \varphi^*(t_n) \in \underline{\mathcal{O}(X)}$ or $\underline{\mathcal{O}(U)}$, $U \subseteq X$

$F_j(\varphi^*(t_1), -, \varphi^*(t_n))$

2) $U \subseteq Y$, $f \in \mathcal{O}(U)$ $\qquad f|_{U_p} = \dfrac{F(x_1, \dots, x_n)}{G(x_1, \dots, x_n)}$

$$(f|_{U_p} \circ \varphi)(Q) = \frac{F(\varphi(Q))}{G(\varphi(Q))} = \frac{F(\varphi^*(t_1)(Q), \dots, \varphi^*(t_n)(Q))}{G(\dots)} =$$

$Q \in \varphi^{-1}(U_p)$

$$= \underbrace{\frac{F(\varphi^*(t_1), \dots, \varphi^*(t_n))}{G(\varphi^*(t_1), \dots, \varphi^*(t_n))}}_{\text{regular fun on } \underline{\underline{\varphi^{-1}(U_p)}}}(Q)$$

$\Rightarrow \forall Q \in \varphi^{-1}(U)$ ~~we~~ we have an express as regular function $\Rightarrow f \circ \varphi| \in \mathcal{O}(\varphi^{-1}(U))$.

$$\underbrace{\varphi = (\varphi_1, \dots, \varphi_n)}_{X \longrightarrow \mathbb{A}^n} \qquad \varphi_1, \dots, \varphi_n \in \mathcal{O}(X)$$

If $X \subseteq \mathbb{A}^m$ closed in $\mathbb{A}^m$, K alg closed

$\mathcal{O}(X) \simeq K[X]$

$$\varphi : X \longrightarrow Y \qquad \varphi = (\underbrace{\varphi_1 \dots \varphi_n}_{\text{polynom. fun. on } X})$$

$\varphi : X \longrightarrow Y$ regular $\Longrightarrow \varphi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$

Thm. X locally closed, $Y \subseteq \mathbb{A}^n$ closed, K alg closed

$\mathrm{Hom}(X, Y)$ set of regular maps $X \to Y$

$\mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$ set of K-homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)$

$$\mathrm{Hom}(X, Y) \xrightarrow{\ *\ } \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$$
$$f \longrightarrow \varphi^*(f)$$

Then * is a bijection: $\exists$ a map $\#$

$\mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X)) \xrightarrow{\ \#\ } \mathrm{Hom}(X, Y)$ which inverts the construction of the comorphism.

Pf: $u \in Hom(\mathcal{O}(Y), \mathcal{O}(X))$ $\quad u: \mathcal{O}(Y) \to \mathcal{O}(X)$ $\quad$ K-homom.

We want $u^{\#}: X \to Y$ s.t. $(u^{\#})^* = u$

If we have $\varphi: X \to Y$ s.t. $\varphi^* = u$

$f \in \mathcal{O}(Y)$ $\quad \boxed{\varphi^*(f) = f \circ \varphi}$

$Y \subseteq \mathbb{A}^n$ $\quad \mathcal{O}(Y) \simeq K[Y] = K[t_1, \dots, t_n]$

$(\varphi^*(t_1), \dots, \varphi^*(t_n)) = \varphi$

$u(t_1), \dots, u(t_n) \in \mathcal{O}(X)$ $\quad$ Def. $u^{\#} = (u(t_1), \dots, u(t_n))$

$u^{\#}: X \longrightarrow \mathbb{A}^n$ regular

We have to check $u^{\#}(X) \subseteq Y$ $\quad Y = V(F_1, \dots, F_r)$

$P \in X$ $\quad u^{\#}(P) = (u(t_1)(P), \dots, u(t_n)(P))$

$F_i(u(t_1)(P), \dots, u(t_n)(P)) = \underbrace{F_i(u(t_1), \dots, u(t_n))}_{\mathcal{O}(X)}(P) =$

u is K-homom.

$= u(F_i(t_1, \dots, t_n))(P) = u(0)(P) = 0(P) = 0$

$F_1 = x_1^2 + x_2 x_3 x_4 \qquad F_1(t_1 \dots t_n) = t_1^2 + t_2 t_3 t_4$

$F_1(u(t_1), \dots, u(t_n)) = u(t_1^2) + u(t_2) u(t_3) u(t_4) =$

$= u(t_1^2 + t_2 t_3 t_4) = u(F_1(t_1, \dots, t_n))$

$u^{\#}: X \longrightarrow Y$ regular

$(u^{\#})^* = u \qquad \mathcal{O}(Y) = K[t_1 \dots t_n]$

$(u^{\#})^*(t_i) = t_i \circ u^{\#} = u(t_i)$

$\varphi: X \to Y$ regular

$(\varphi^*)^{\#} = \varphi$

$$\mathrm{Hom}(X,Y) \underset{\#}{\overset{*}{\longleftrightarrow}} \mathrm{Hom}(\mathcal{O}(Y),\mathcal{O}(X))$$

Y closed in $\mathbb{A}^n$ $\qquad \mathcal{O}(Y) \simeq \mathcal{K}[Y] = K[t_1, \dots, t_n]$

The construction of $\#$ is functorial:

1) $1_{\mathcal{O}(Y)} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(Y)$ $\qquad$ Y affine

$$1^{\#}_{\mathcal{O}(Y)} = (1_{\mathcal{O}(Y)}(t_1), \dots, 1_{\mathcal{O}(Y)}(t_n)) = (t_1 \dots t_n) = 1_Y$$

2)
$$\begin{tikzcd} \mathcal{O}(Z) \arrow[r, "v"] \arrow[rr, bend right, "u\circ v"'] & \mathcal{O}(Y) \arrow[r, "u"] & \mathcal{O}(X) \end{tikzcd}$$

$Z \subseteq \mathbb{A}^m$ closed

$Y \subseteq \mathbb{A}^n$ "

$$\begin{tikzcd} X \arrow[r, "u^{\#}"] \arrow[rr, bend right, "(u\circ v)^{\#}"'] & Y \arrow[r, "v^{\#}"] & Z \end{tikzcd}$$

$$\boxed{v^{\#} \circ u^{\#} = (u \circ v)^{\#}}$$

Consequence $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ closed, K alg. closed

$$\boxed{X \simeq Y \text{ if and only if } \mathcal{O}(X) \simeq \mathcal{O}(Y)}$$

$\Longrightarrow$

$\Longleftarrow$ functionality of $\#$

$\underline{\mathcal{O}(\mathbb{A}^1)} = K[x] = K[x, x^2, x^3] = \frac{K[x,y,z]}{\langle y-x^2, z-x^3\rangle} = \mathcal{O}(X)$

x a variable

$\overline{X \text{ skew cubic}}$

$\Longrightarrow \mathbb{A}^1 \simeq X$

$u : K[x, x^2, x^3] \longrightarrow K[x]$

$F(x, x^2, x^3) \longrightarrow F(x)$

$u^\# : \mathbb{A}^1 \longrightarrow X$

$t \longrightarrow u^\#(t) = (u(x)t), u(x^2)(t), u(x^3)(t))$

$= (t, t^2, t^3)$

$K[x] = K[x, x^2, \ldots, x^n] = \mathcal{O}(X_n)$

$X_n = \{(t, t^2, \ldots, t^n) \mid t \in K\}$ $\quad I(X_n) = (x_2 - x_1^2, x_3 - x_1^3, \ldots)$

$X_n \simeq \mathbb{A}^1$
$\cap$
$\mathbb{A}^n$

rational normal curve of degree n

$\varphi : X \to Y$ regular, Y irreducible $\boxed{f \in K(Y)}$

$\varphi^*(f)$? Is it possible to consider a comorphism $K(Y) \longrightarrow K(X)$?

$\underline{f = \text{germ of a pair}}$ (U, f) $\varphi^*(f) \in \varphi^{-1}(U)$

If $\boxed{\operatorname{dom} f \cap \varphi(X) = \emptyset}$ $\underline{\varphi^*(f) \text{ does not exist}}$

$X \xrightarrow{\varphi} \varphi(X) \subseteq Y$

$\operatorname{dom} f \xrightarrow{f} K$

$\varphi^{-1}(\operatorname{dom} f) = \emptyset$

To def. $\varphi^* : K(Y) \longrightarrow K(X)$ we need to be sure that $\forall f \in K(Y)$ $\operatorname{dom} f \cap \varphi(X) \neq \emptyset$

Assumption: φ is dominant $\boxed{\overline{\varphi(X)} = Y}$

If φ is domin., $\underset{\neq \emptyset}{U} \subseteq Y$ open $\Longrightarrow$ $\varphi(X) \cap U \neq \emptyset$

In partic. if $U = \operatorname{dom} f$, $f \in K(X)$ $\Longrightarrow$ $\varphi(X) \cap \operatorname{dom} f \neq \emptyset$

If φ is regular and dominant $\Rightarrow$ $\varphi^*: K(Y) \longrightarrow K(X)$

$\langle U, f\rangle \longrightarrow \langle \varphi^{-1}(U), f\circ\varphi| \rangle$

Now φ^* is a field K-homom. : injective

$\Rightarrow K(Y) \simeq \varphi^* K(Y) \subseteq K(X)$

$\Rightarrow \operatorname{tr.d.} K(X)/K \geq \operatorname{tr.d.} K(Y)/K$

$\operatorname{tr.d.} K(X)/K = \dim X$, $\quad \operatorname{tr.d.} K(Y)/K = \dim Y$

If $\exists$ $X \longrightarrow Y$ reg. dominant $\Rightarrow$ $\dim X \geq \dim Y$

$\forall y \in Y$ $\quad \varphi^{-1}(y)$

$\exists U \subseteq Y$ open $\neq \emptyset$: $\boxed{\forall y \in U}$ $\dim \varphi^{-1}(y) = \dim X - \dim Y$

$\varphi : X \to Y$ regular, domin., Y irred.

$P \in X$, $\varphi(P) = Q$ $\quad \varphi^* : \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$

$$\begin{array}{ccccc} & & P & & Q \\ & & X & \longmapsto & Y \\ & & & & \cup \\ \varphi^{-1}(U) & \longrightarrow & & & U \longrightarrow K \\ P & & & & Q \end{array} \qquad \begin{array}{ccc} \mathcal{O}_{Y,Q} & \longrightarrow & \mathcal{O}_{X,P} \\ \ni & & \ni \\ f & \longmapsto & \varphi^*(f) \end{array}$$

is a <u>local</u> homom.

$f: A \to B$ $\quad f$ homom. of local rings

Def. f is a local homom. if $f(m_A) \subseteq m_B \iff f^{-1}(m_B) = m_A$

"$\Rightarrow$" $f(m_A) \subseteq m_B$

$a \in A$ $\quad f(a) \in m_B$, $f(a)$ is not invertible

If $\exists\, a'$ $\quad aa' = 1$ $\quad f(a)f(a') = f(aa') = f(1) = 1$: contrad.

$\Rightarrow a \in m_A$

"$\Leftarrow$" $f^{-1}(m_B) = m_A$, $\quad a \in m_A$ $\quad$ ~~$f(a)$ not invertible~~ $\quad f^{-1}(m_B)$

~~If $f(a)$ is inv., $\exists\, b \in B$ $f(a)b = 1$~~

~~b inv.~~ $\quad f(a) \in m_B$

$\varphi : X \longrightarrow Y$

$\varphi^* : \mathcal{O}_{Y,Q} \longrightarrow \mathcal{O}_{X,P}$ $\quad Q = \varphi(P)$

$\varphi\left(f(Q)\right) = 0 \qquad \varphi^*(f) \in \mathfrak{m}_{X,P}$