Multiplication of distribution with a smooth functions

def. Let Ω open set in $\mathbb{R}^d$

$$\left(\partial_{x_j}^{(d)}(f\cdot T)\right)(\varphi) \underset{\text{def of diff.}}{=} -(f\cdot T)(\partial_{x_j}\varphi) \underset{\text{def. of mult.}}{=} T(f\cdot\varphi)$$

usual multiplication

$$= -T(f\cdot\partial_{x_j}\varphi)$$

Pr.

$$= -T\left(\partial_{x_j}(f\cdot\varphi) - (\partial_{x_j} f)\cdot\varphi\right)$$

order m

f · T has order $\le m$)

proof. 1) $f\cdot T$ is linear

$$(f\cdot T)(a\varphi + b\psi) = T(f\cdot(a\varphi+b\psi)) = T(a(f\varphi)+b(f\psi)) = a\,T(f\varphi) + b\,T(f\psi) = a(f\cdot T)(\varphi) + b(fT)(\psi)$$

2) $f\cdot T$ satisfies the inequality of distributions.

T is distribution

we know that $\forall K$ compact in Ω, $\exists C_K > 0$, $\exists m_K \in \mathbb{N}$

s.t. $$|T(\varphi)| \le C_K \cdot \sum_{|\alpha|\le m_K} \sup_{x\in\Omega} |\partial^\alpha \varphi| \qquad (*)$$

for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi \subseteq K$

Take K

$\mathcal{D}(\Omega)$ with supp $\subseteq K$

$$|f\cdot T(\varphi)| = |T(f\varphi)| \underset{(*)}{\le} C_K \cdot \sum_{|\alpha|\le m_K} \sup_{x\in\Omega} |\partial^\alpha(f\varphi)| \qquad \forall \varphi \in \mathcal{D}(\Omega)$$

$$\partial^\alpha(f\varphi) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f\, \partial^\beta\varphi \qquad \text{(Liebnitz)}$$

$$\sup_{x\in\Omega} |\partial^\alpha(f\varphi)| \le \sup_{x\in\Omega}\left(\left|\sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} f\, \partial^\beta\varphi\right|\right)$$

$$\le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \sup_{x\in K} |\partial^{\alpha-\beta} f| \cdot \sup_{x\in\Omega} |\partial^\beta\varphi|$$

$$\sum_{|\alpha|\le m_K} \sup_{x\in\Omega} |\partial^\alpha(f\varphi)| \le \tilde{C} \sum_{|\beta|\le m_K} \sup_{x\in\Omega} |\partial^\beta\varphi|$$

this const. depends on $\sup_{\substack{x\in K \\ |\beta|\le m_K}} |\partial^\beta f|$

finally

$$|fT(\varphi)| \le \tilde{C}_K \sum_{|\beta|\le m_K} \sup_{x\in\Omega} |\partial^\beta\varphi| \qquad \forall\varphi\in\mathcal{D}(\Omega),\ \operatorname{supp}\varphi\subseteq K$$

$$\tilde{C}_K = C_K\cdot\tilde{C}$$

Ex. let $f \in \mathcal{E}^\infty(\Omega)$, let $T \in \mathcal{D}'(\Omega)$

$$\underbrace{\partial_{x_j}(f\cdot T)}_{\text{diff. in the sense of distribution}} = \partial_{x_j}^{(d)}(f\cdot T)$$

$$\left(\partial_{x_j}^{(d)}(f\cdot T)\right)(\varphi) \underset{\text{def of diff.}}{=} -(f\cdot T)(\partial_{x_j}\varphi) \underset{\text{def. of mult.}}{=}$$

cancel

$$= -T(f\cdot\partial_{x_j}\varphi)$$

$$= -T\left(\partial_{x_j}(f\cdot\varphi) - (\partial_{x_j} f)\cdot\varphi\right)$$

$$\left(\partial^{(d)}_{x_j}(f\cdot T)\right)(\varphi) \underset{\substack{\uparrow \\ \text{def of diff.}}}{=} -(f\cdot T)(\partial_{x_j}\varphi) \underset{\substack{| \\ \text{def. of mult.}}}{=}$$

$$= -T(f\cdot\partial_{x_j}\varphi)$$

$$= -T\left(\partial_{x_j}(f\cdot\varphi) - (\partial_{x_j}f)\cdot\varphi\right)$$

$$= -T(\partial_{x_j}(f\varphi)) + T((\partial_{x_j}f)\varphi)$$

$$= (\partial^{(d)}_{x_j}T)(f\varphi) + (\partial_{x_j}f\cdot T)(\varphi)$$

$$= (f\cdot\partial^{(d)}_{x_j}T)(\varphi) + (\partial_{x_j}f\cdot T)(\varphi)$$

$$= (f\cdot\partial^{(d)}_{x_j}T + \partial_{x_j}f\cdot T)(\varphi)$$

Conclusion

$$\boxed{\partial^{(d)}_{x_j}(f\cdot T) = f\cdot\partial^{(d)}_{x_j}T + \underbrace{\partial_{x_j}f}_{\text{classical}}\cdot T}$$

Th. (Du Bois Reymond)

Let $f, g \in \mathcal{C}(\Omega)$ then consider $T_f, T_g \in \mathcal{D}'(\Omega)$

$(f, g \in L^1_{loc}(\Omega))$

Suppose that $\partial^{(d)}_{x_j} T_f = T_g$

then f is differentiable w.r.t. x_j in classical sense

and $\partial_{x_j} f = g$

proof. step 1. suppose that $f, g \in \mathcal{C}_0(\Omega)$ (f, g continuous with compact support)

let $(\rho_n)_n$ be a mollifier

$(\rho_n(x) = n^d \rho(nx)$, $\rho \in \mathcal{C}^\infty_0(\mathbb{R}^d)$, $\rho \geq 0$, supp $\rho \subseteq B(0,1)$, $\int \rho(x)dx = 1$)

for $n \geq \bar{n}$ we have that

$f * \rho_n$, $g * \rho_n$ are in $\mathcal{C}^\infty_0(\Omega)$

and $f * \rho_n \longrightarrow f$ uniformly

$g * \rho_n \longrightarrow g$ uniformly

$$(f * \rho_n)(x) = \int_{\mathbb{R}^d} f(y)\rho_n(x-y)\,dy = \int_\Omega f(y)\underbrace{\rho_n(x-y)}\,dy$$

I can think to this funct. as x is a parameter

$y \longmapsto \rho_n(x-y)$

is a test function in y

$= T_f(\rho_n(x - \cdot))$

similarly $(g * \rho_n)(x) = T_g(\rho_n(x - \cdot))$

so $\partial_{x_j}(f * \rho_n)(x) = (f * \partial_{x_j}\rho_n)(x)$

$= T_f(\partial_{x_j}(\rho_n(x-\cdot))) = T_f(-\partial_{y_j}(\rho_n(x - \cdot)))$

$= (\partial^{(d)}_{y_j} T)(\rho_n(x - \cdot))$

so $\partial_{x_j}(f * \rho_n)(x) = (\partial^{(d)}_{y_j} T)(\rho_n(x - \cdot)) = T_g(\rho_n(x - \cdot))$

now I apply the hypothesis

$= (g * \rho_n)(x)$

conclusion

$f * \rho_n \longrightarrow f$ uniformly

$\partial_{x_j}(f * \rho_n) \longrightarrow g$ uniformly

$\parallel$

$g * \rho_n$

I can apply the classical result on uniform convergence step 2. $f, g \in \mathcal{C}(\Omega)$

take $x_0 \in \Omega$

consider $\chi \in \mathcal{C}_0^\infty(\Omega)$ st. $\chi = 1$ in a nbh of x_0

consider $\chi \cdot T_f$

$$\chi \cdot T_f(\varphi) = T_f(\chi\varphi) = \int_\Omega f \chi \varphi = \int_\Omega (f\chi)\varphi = T_{\chi f}(\varphi)$$

so $\chi \cdot T_f = T_{\chi f}$

compute $\partial_{x_j}^{(d)}(\chi T_f) = \partial_{x_j}\chi \cdot T_f + \chi \cdot \underbrace{\partial_{x_j}^{(d)} T_f}_{\text{use the hypothesis}}$

$$= \partial_{x_j}\chi \cdot T_f + \chi \cdot T_g$$

$$= T_{\partial_{x_j}\chi \cdot f + \chi \cdot g}$$

I have χf, $\chi g + \partial_{x_j}\chi \cdot f \in \mathcal{C}_0(\Omega)$

and $\partial_x^{(d)} T_{\chi f} = T_{\chi g + \partial_{x_j}\chi f}$

I can apply step 1

so $\chi \cdot f$ is differentiable in classical sense

and $\partial_{x_j}(\chi f) = \chi g + \partial_{x_j}\chi \cdot f$

in the nbh of x_0 in which $\chi = 1$

we have $\partial_{x_j} f = g$

QED

Distributions with compact support

Rem. Let Ω open set

one $\underbrace{\mathcal{D}(\Omega)} = C_0^\infty(\Omega)$

there is a complicate topology
but it is of interest only the
set of "converging sequences".

$(\varphi_n)_n$ in $\mathcal{D}(\Omega)$ converges to φ "in the sense of $\mathcal{D}$"

if $\underbrace{(\varphi_n - \varphi)}_{\psi_n} \longrightarrow 0$ "in the sense of $\mathcal{D}(\Omega)$"

i.e.
$\exists K$ compact s.t. $\operatorname{supp} \psi_n \subseteq K$ for all n and

$\forall \alpha$, $\partial^\alpha \psi_n \longrightarrow 0$ uniformly

Consider $\mathcal{C}^{\infty}(\Omega)$

¿ which topology ?

consider $\Omega_0 \subseteq \Omega_1 \subseteq \dots \subseteq \Omega_n \subseteq \dots$

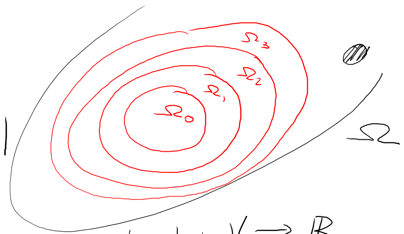
Ω_j open set, $\overline{\Omega_j}$ compact, $\overline{\Omega_j} \subseteq \Omega_{j+1}$

and $\Omega = \bigcup_{j=0}^{+\infty} \Omega_j$

for $f \in \mathcal{C}^{\infty}(\Omega)$

define the following

$$\tilde{p}_j(f) = \sum_{|\alpha| \le j} \sup_{x \in \Omega_j} |\partial^{\alpha} f(x)|$$



$\tilde{p}_j$ is a seminorm of $\mathcal{C}^{\infty}(\Omega)$

$p : V \to \mathbb{R}$, V vector space

s.t.

seminorm:
$p(\lambda v) = |\lambda| v$
$p(v+w) \le p(v) + p(w)$
and $p(0) = 0$

if $p(v) = 0 \Rightarrow v = 0$ norm

distance on $\mathcal{C}^{\infty}(\Omega)$

consider $d(f,g)$

$$= \sum_{j=0}^{+\infty} \frac{1}{2^j} \cdot \frac{\tilde{p}_j(f-g)}{\tilde{p}_j(f-g)+1}$$

it is possible to prove that $\mathcal{C}^{\infty}(\Omega)$ with d is a complete metric space (Fréchet space)

it is possible to prove the following two properties

1) Let $S : \mathcal{C}^{\infty}(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) linear

S is continuous if and only if

$\exists\, j \in \mathbb{N}, \exists\, C > 0$ s.t.

$$|S(f)| \le C\, \tilde{p}_j(f)$$

2) if $(f_n)_n$ is a sequence in $\mathcal{C}^{\infty}(\Omega)$

$f_n \to 0$ in the metric d

if and only if

$f_n \to 0$ for all the seminorms.

rem, we have $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$

we have $\mathcal{E}(\Omega) = C^\infty(\Omega)$

we have $\mathcal{D}(\Omega) \subseteq \mathcal{E}(\Omega)$

we consider $\mathcal{D}'(\Omega)$ $\mathcal{E}'(\Omega)$

$\mathcal{D}'(\Omega)$ = distributions; $\mathcal{E}'(\Omega)$ = dual space of $C^\infty(\Omega)$

if $S \in \mathcal{E}'(\Omega)$

what about $S_{|\mathcal{D}(\Omega)}$?

Th 1) if $S \in \mathcal{E}'(\Omega)$ then $S_{|\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$ and supp $S_{|\mathcal{D}(\Omega)}$ is compact.

Th 2). if $T \in \mathcal{D}'(\Omega)$ and supp T is compact then there exist a unique $S \in \mathcal{E}'(\Omega)$ s.t. $S_{|\mathcal{D}(\Omega)} = T$

$\mathcal{E}' \subseteq \mathcal{D}'$

$\mathcal{E} \supseteq \mathcal{D}$

proof (Th 1)

$S \in \mathcal{E}'(\Omega)$

S is linear and $\exists C>0, \exists \bar{j}$ s.t.

$$|S(f)| \le C \cdot \tilde{p}_{\bar{j}}(f)$$

i.e.

$$|S(f)| \le C \sum_{|\alpha| \le \bar{j}} \sup_{x \in \overline{\Omega_{\bar{j}}}} |\partial^\alpha f(x)| \quad \forall f \in \mathcal{C}^\infty(\Omega)$$

consider $S|_{\mathcal{D}(\Omega)}$

$\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\quad |S(\varphi)| \le C \sum_{|\alpha| \le \bar{j}} \sup_{x \in \overline{\Omega_{\bar{j}}}} |\partial^\alpha \varphi(x)| \quad (*)$

fix K compact set in Ω choose $C_K = C$, $m_K = \bar{j}$

we have $|S(\varphi)| \le C_K \cdot \sum_{|\alpha| \le m_K} \sup_{x \in \Omega} |\partial^\alpha \varphi| \quad \forall \varphi \in \mathcal{C}_0^\infty(\Omega)$ with supp in K

$S|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$

if φ has support outside $\overline{\Omega_{\bar{j}}}$ then $(*)$ $S(\varphi) = 0$

so that $\operatorname{supp}(S|_{\mathcal{D}(\Omega)}) \subseteq \overline{\Omega_{\bar{j}}}$

proof (Th. 2) (take $T \in \mathcal{D}'(\Omega)$ with supp T compact
prove that $\exists!\ S \in \mathcal{E}'(\Omega)$ s.t. $S_{|\mathcal{D}(\Omega)} = T$)

so $T \in \mathcal{D}'(\Omega)$
with compact support.

consider $\chi \in \mathcal{C}_0^\infty(\Omega)$ $\chi = 1$ in a nbhd of the support of T

define $S \in \mathcal{E}'(\Omega)$
in this way $S(f) = T(\chi f)$

I prove that S is in $\mathcal{E}'(\Omega)$ and it is unique.

<u>uniqueness</u> suppose χ_1 and χ_2 two $\mathcal{C}_0^\infty$ fct which are 1 in a nbhd of supp T

then $f \in \mathcal{C}^\infty(\Omega)$, $T((\chi_1 - \chi_2) f) = 0$ because

$(\chi_1 - \chi_2) \cdot f \equiv 0$ in a nbhd of supp of T

so that $T(\chi_1 f) = T(\chi_2 f)$

$|S(f)| = |T(\chi f)|$

T is a distribution consider $\Omega_{\bar{J}}$ s.t. $\Omega_{\bar{J}} \supseteq$ supp T

consider $\overline{\Omega_{\bar{J}}} = K$ use the def. of T with this K

$\exists C_K, \exists m_K$ s.t.

$$|T(\varphi)| \le C_K \cdot \sum_{|\alpha| \le m_K} \sup_{x \in \Omega} |\partial^\alpha \varphi|$$ for all φ with supp in $\overline{\Omega_{\bar{J}}}$

I take $\chi \in \mathcal{C}_0^\infty(\Omega)$
s.t. $\chi = 1$ on a nbhd of supp T
and supp $\chi \subseteq \overline{\Omega}_{\bar{J}}$

$|S(f)| = |T(\chi f)| \le$

$\chi f \in \mathcal{C}_0^\infty$ with supp in $\overline{\Omega_{\bar{J}}}$ so that

$$\le C_K \sum_{|\alpha| \le m_K} \sup_{x \in \Omega} |\partial^\alpha \chi f|$$

$$\le C_K \sum_{|\alpha| \le m_K} \sup_{x \in \Omega_J} |\partial^\alpha f|$$

I can take $\bar{J}$

$$\le C_K\, \tilde{p}_J(f)$$