Exercises 4

key words: Topological vector spaces with a topology defined by a countable number of seminorms, spaces $C^m(\Omega)$, $C^\infty(\Omega)$. Inductive limit topology, spaces $C_0^m(\Omega)$, $C_0^\infty(\Omega)$. Convolution. Mollifiers.

1) (Borel's theorem) Let $(a_n)_n$ be a sequence in $\mathbb{R}$. Let $\varphi \in \mathcal{D}(]-2,2[)$ such that $\varphi(x) = 1$ for $|x| \leq 1$.

- Show that there exists a sequence $(\lambda_k)_k$ in $\mathbb{R}$ such that, if we set
$$f_k(x) = \frac{a_k}{k!} x^k \varphi(\lambda_k x),$$
then
$$\sup_{x\in\mathbb{R}} |f_k^{(j)}(x)| \leq 2^{-k}, \quad \text{for all } 0 \leq j \leq k-1.$$
- Deduce that the series $\sum_k f_k(x)$ defines a function $f(x) \in C^\infty$ such that, for all j, $f^{(j)}(0) = a_j$.

2) Let Ω be an open set in $\mathbb{R}^n$. Let m and k be two positive integers, with $k \geq m$. Let $P(x, \partial_x) = \sum_{|\alpha|=m} a_\alpha(x)\partial_x^\alpha$ be a differential operator with $a_\alpha \in C^{k-m}(\Omega)$.

- Prove that $P(x, \partial_x)$ is continuous from $C^k(\Omega)$ to $C^{k-m}(\Omega)$.

3) Show that there exists no function $\delta \in C_0^0(\mathbb{R})$ such that $\delta * f = f$ for all $f \in C_0^0(\mathbb{R})$.

4) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $h \in \mathbb{R}^n \setminus \{0\}$. For all $n \in \mathbb{N}$, we set
$$\varphi_n(x) = n(\varphi(x + \frac{h}{n}) - \varphi(x)).$$

- Prove that $(\varphi_n)_n$ converges, in the sense of $\mathcal{D}$, to a function to be determined.

5) (Poincaré inequality) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, Ω an open bounded set in $\mathbb{R}^n$.

- Prove that, for $i = 1, 2, \ldots, n$,
$$\int_{\mathbb{R}^n} |\varphi(x)|^2 \, dx = -2 \int_{\mathbb{R}^n} x_i \varphi(x) \partial_{x_i} \varphi(x) \, dx.$$
- Prove that there exist $C > 0$ such that, for all $\psi \in \mathcal{D}(\Omega)$,
$$\int_\Omega |\psi(x)|^2 \, dx \leq C \sum_{i=1}^n \int_\Omega |\partial_{x_i} \psi(x)|^2 \, dx.$$

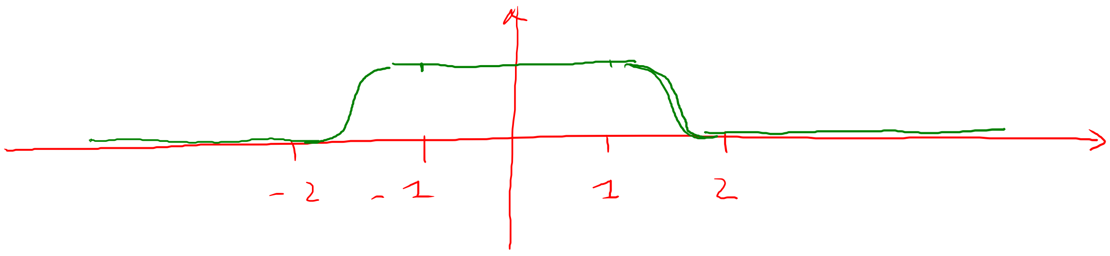
6) Costruct a sequence $(\varphi_k)_k$ in $\mathcal{D}(\mathbb{R})$ such that

- for each point $x \in \mathbb{R}$, the sum $\sum_k \varphi_k(x)$ is a finite sum;
- for each point $x \in \mathbb{R}$, $\sum_k \varphi_k(x) = 1$.

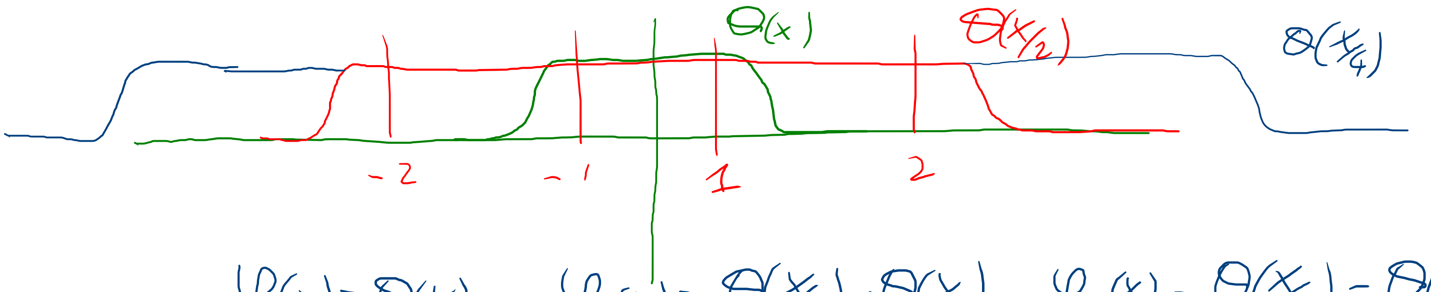
6) Costruct a sequence $(\varphi_k)_k$ in $\mathcal{D}(\mathbb{R})$ such that

- for each point $x \in \mathbb{R}$, the sum $\sum_k \varphi_k(x)$ is a finite sum;
- for each point $x \in \mathbb{R}$, $\sum_k \varphi_k(x) = 1$.

$$\theta(x) = \begin{cases} 0 & \text{if } |x| > \frac{19}{10} \\ 1 & \text{if } |x| < \frac{11}{10} \end{cases}$$



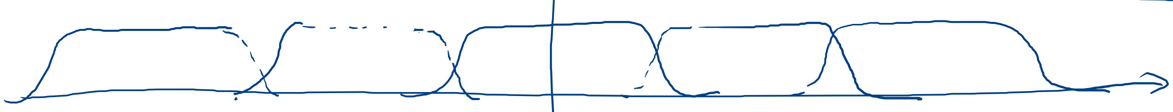
$\theta(x)$, $\theta(x/2)$, $\theta(x/4)$, ..., $\theta\left(\frac{x}{2^n}\right)$



$\varphi_1(x) = \theta(x)$, $\varphi_2(x) = \theta(x/2) - \theta(x)$, $\varphi_3(x) = \theta(x/4) - \theta(x/2)$

$$\varphi_n(x) = \theta\left(\frac{x}{2^{n-1}}\right) - \theta\left(\frac{x}{2^{n-2}}\right)$$

"dyadic decomposition"



$$u(x) \xrightarrow{\mathcal{F}} \hat{u}(\xi) \longrightarrow \underbrace{\varphi_n(\xi)\hat{u}(\xi)}_{} \xrightarrow{\mathcal{F}^{-1}} \Delta_n u$$

Fourier transform

part of u which has "frequence" in all annules

$\Delta_n u$

4) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $h \in \mathbb{R}^n \setminus \{0\}$. For all $n \in \mathbb{N}$, we set

$$\varphi_n'(x) = n(\varphi'(x + \frac{h}{n}) - \varphi'(x)).$$

- Prove that $(\varphi_n)_n$ converges, in the sense of $\mathcal{D}$, to a function to be determined.

$T_{\varphi_n}(\psi) \to T_{h\varphi'}(\psi)$

$\int \varphi_n(\psi)$

$\mathcal{D}'$

$\psi = h \cdot \nabla \varphi$

convergence in the sense of $\mathcal{D}$ means that $\exists K$ compact s.t. $\forall n$, $\operatorname{supp} \varphi_n \subseteq K$

and $\varphi_n \to \psi$ (with $\psi \in \mathcal{D}$) uniformly

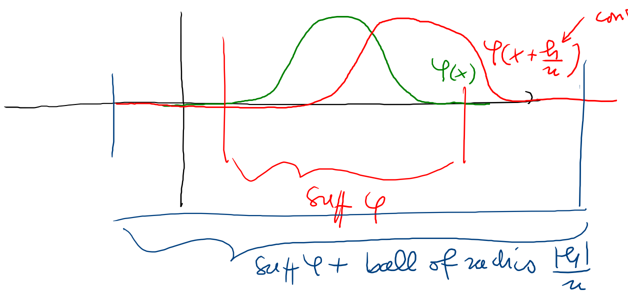
and $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly.

a) $\forall n$, $\varphi_n(x) = n\left(\varphi(x + \frac{h}{n}) - \varphi(x)\right)$

$\varphi_n \in \mathcal{D}(\mathbb{R}^n)$

$\operatorname{supp} \varphi_n \subseteq \operatorname{supp} \varphi + B(0, \frac{|h|}{n})$ compact $\underline{\underline{1}}$ compact $\underline{\underline{K}}$

in 1D



in 1D

$$n\left(\varphi(x + \frac{h}{n}) - \varphi(x)\right) = \frac{\varphi(x + \frac{h}{n}) - \varphi(x)}{\frac{1}{n}} = h\underbrace{\left(\frac{\varphi(x + h/n) - \varphi(x)}{\frac{h}{n}}\right)}_{\downarrow n \to +\infty \; \varphi'(x)}$$

$$\lim_{n \to +\infty} n\left(\varphi(x + h/n) - \varphi(x)\right) = h\varphi'(x)$$

prove that this is a uniform convergence

$$\underbrace{\lim_{n \to +\infty} \left(\sup_{x \in \mathbb{R}} \left| n\underbrace{\left(\varphi(x + \frac{h}{n}) - \varphi(x)\right)}_{h\varphi'(\theta_x) \text{ with } \theta_x \in [x, x + \frac{h}{n}]} - h\varphi'(x) \right|\right)}_{} = 0$$

$$\leq |h| \cdot \underbrace{\sup_{\substack{x \in \mathbb{R} \\ \theta_x \in [x, x + \frac{h}{n}]}} |\varphi'(\theta_x) - \varphi'(x)|}_{\leq \|\varphi''\|_{L^\infty} \cdot \frac{|h|}{n}} = 0$$

3) Show that there exists no function $\delta \in C_0^0(\mathbb{R})$ such that $\delta * f = f$ for all $f \in C_0^0(\mathbb{R})$.

take $f = \rho_n$ where $\rho_n(x) = n\rho(nx)$

↑ usual function

$\delta * \rho_n$

$\rho \in C_0^\infty(\mathbb{R})$, supp $\rho \subseteq [-1,1]$

$\rho \geq 0$ $\int \rho = 1$

we know that $\delta * \rho_n \to \delta$ uniformly

but $\delta * \rho_n = \rho_n$ by assumption

so that $\rho_n \to \delta$ uniformly impossible

since $\rho_n(0) = n \longrightarrow +\infty$

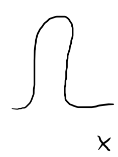
$\delta * f = \int \delta(y) f(x-y)\, dy$

let $\delta \in C_0^0(\mathbb{R})$, suppose $\delta(x_0) \neq 0$, $\delta(x_0) > 0$, note $x_0 \neq 0$

so $\exists\ \varepsilon > 0$ s.t. $\delta(x) \geq c > 0$ for $x \in [x_0-\varepsilon, x_0+\varepsilon]$

then $\int \delta(y)\, f(x-y)$ I can choose f s.t. $f = \rho_n$

$\rho_n(x) = \int \delta(y)\, \rho_n(x-y)\, dy$



↑ by assumption

$\rho_n(x) = 0$ in x_0

if n is sufficiently big

is not 0 because

given $(a_n)_n$, $\exists f \in \mathcal{C}^\infty(\mathbb{R})$ s.t.

$\forall n$, $f^{(n)}(0) = a_n$.

this is connected with the topic of analytic functions

Analytic function does not satisfies this property

$$\boxed{f(x) = \underbrace{\sum_n \frac{f^{(n)}(0)}{n!} x^n}_{\text{Taylor series}}}$$

(in a nbh. of 0)

Es. 3. Topology of $\mathcal{E}^m(\Omega)$

$\Omega_0 \subseteq \Omega_1 \subseteq \dots \subseteq \Omega_j \subseteq \dots$ $\quad$ Ω_j open, $\overline{\Omega}_j$ compact, $\bigcup_j \Omega_j = \Omega$

$$p_j(f) \doteq \sum_{|\alpha|\le m} \sup_{x\in\Omega_j} |\partial^\alpha f|$$

nbh of 0 are sets that contain a semi-ball

U is a nbh of 0 if

$\Rightarrow \exists\, \varepsilon>0, \exists j$ s.t. $U \supseteq \{f\in \mathcal{E}^m(\Omega) : p_j(f)<\varepsilon\}$

(metric $d(f,g) = \sum_{j=0}^{+\infty} \frac{1}{2^j} \frac{p_j(f-g)}{1+p_j(f-g)}$)

$\mathcal{E}^m(\Omega)$ is a Fréchet space

to solve the ex. it's sufficient to prove that

$f \longmapsto af$ is cont. from $\mathcal{E}^m(\Omega) \to \mathcal{E}^m(\Omega)$

$f \longmapsto \partial^\alpha f$ is cont. from $\mathcal{E}^m \to \mathcal{E}^{m-k}$, $|\alpha|=k$

if $L: V \longrightarrow W$ $\quad$ V, W normed, L linear

cont. means $\exists C$: $\|Lf\|_W \le C\|f\|_V$

for Fréchet similar

$V \xrightarrow{L} W$ $\quad$ L continuous $\forall j, \exists i, \exists C$: $q_j(Lf) \le C\, p_i(f)$

family $(p_j)_j$ seminorms $\quad$ family $(q_j)_j$ seminorms

$\mathcal{E}^m(\Omega) \xrightarrow{\partial^\alpha} \underbrace{\mathcal{E}^{m-k}(\Omega)}_{\text{seminorm here}}$ $\quad |\alpha|=k$

$\sup_{x\in\Omega_j} \sum_{|\beta|\le m-k} \partial^\beta(\partial^\alpha f)$

$$\underbrace{\sup_{x\in\Omega_j} \sum_{|\beta|\le m-k} |\partial^\beta(\partial^\alpha f)|}\le 1\cdot \sup_{x\in\Omega} \sum_{|\gamma|\le m} |\partial^\gamma f(x)|$$

$$\boxed{p_j^{m-k}(\partial^\alpha f) \le p_j^m(f)}$$ continuity

$a: \mathcal{E}^m(\Omega) \longrightarrow \mathcal{E}^m(\Omega)$

$f \longmapsto af$

¿ $p_j^m(af) \le C\, p_{\tilde{j}}^m(f)$?

$p_j^m(af) = \sup_{x\in\Omega_j} \sum_{|\alpha|\le m} (\partial^\alpha(af))$

$\sup_{x\in\overline{\Omega}_j} \partial^\alpha(af) \le$

$$\partial^\alpha(af) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} a\, \partial^\beta f$$

$$\sup_{x\in\overline{\Omega}_j} |\partial^\alpha (af)| \le C_j \Big(\sum_{|\beta|\le m} \sup|\partial^\beta a|\Big)\Big(\sum_{|\beta|\le m} \sup|\partial^\beta f|\Big)$$

cont. dep only on m