April 26th

distribution with compact support

<u>Th</u>. Let Ω be an open set in $\mathbb{R}^d$

suppose $S \in \mathcal{E}'(\Omega) \leftarrow$ dual space of $\mathcal{E}(\Omega) = C^\infty(\Omega)$

then $S|_{\mathcal{D}(\Omega)} \in \mathcal{D}'(\Omega)$ and $\operatorname{supp}(S|_{\mathcal{D}(\Omega)})$ is a compact set

Conversely if $T \in \mathcal{D}'(\Omega)$ and $\operatorname{supp} T$ is a compact set in Ω

then $\exists$ a unique S in $\mathcal{E}'(\Omega)$ s.t. $S|_{\mathcal{D}(\Omega)} = T$.

<u>remark</u> We will say that $\mathcal{E}'(\Omega)$ is the space of distribution with compact support

Convolution of distributions

remark: let $f \in L^1_{loc}(\mathbb{R}^d)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$

it makes sense to consider

$$f*\varphi(x) = \int_{\mathbb{R}^d} f(y)\varphi(x-y)\,dy = T_f(\varphi_x) \quad \text{where } \varphi_x(y) = \varphi(x-y)$$

def. let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$

I define $T*\varphi(x) = T(\varphi_x)$ where $\varphi_x(y) = \varphi(x-y)$

remark: f and φ as before

$x \longmapsto f*\varphi(x)$ which regularity?

is a C^∞ function and $\partial_{x_j}(f*\varphi)(x) = (f*\partial_{y_j}\varphi)(x)$

Th. let $T \in \mathcal{D}'(\mathbb{R}^d)$, $\varphi \in C_0^\infty(\mathbb{R}^d)$

then $T*\varphi$ is a C^∞ function

and $\partial_{x_j}(T*\varphi)(x) = (T*\partial_{y_j}\varphi)(x) = (\underbrace{\partial_{y_j}T}_{\text{is the derivative of } T \text{ as a distribution}}*\varphi)(x)$

proof (sketch)

I prove the continuity of $x \longmapsto (T*\varphi)(x)$

I consider $(x_n)_n$ in $\mathbb{R}^d$ s.t. $x_n \xrightarrow{n} \bar{x}$

I prove that $(T*\varphi)(x_n) \longrightarrow (T*\varphi)(\bar{x})$ $\Rightarrow$ continuity

$(T*\varphi)(x_n) = T(\varphi_{x_n})$ where $y \longmapsto \varphi_{x_n}(y) = \varphi(x_n - y)$

$(T*\varphi)(\bar{x}) = T(\varphi_{\bar{x}})$ where $y \longmapsto \varphi_{\bar{x}}(y) = \varphi(\bar{x} - y)$

$\varphi_{x_n}, \varphi_{\bar{x}}$ are $C_0^\infty(\mathbb{R}^d_y)$ and $\exists K$ compact in $\mathbb{R}^d_y$ s.t.

supp φ_{x_n}, supp $\varphi_{\bar{x}} \subseteq K$

It is possible to see that $\varphi_{x_n} \longrightarrow \varphi_{\bar{x}}$ uniformly

and the same for $\partial_y^\alpha(\varphi_{x_n}) \longrightarrow \partial_y^\alpha(\varphi_{\bar{x}})$

so $(\varphi_{x_n} - \varphi_{\bar{x}})_n$ is a sequence in $C_0^\infty(\mathbb{R}^d)$ which goes to 0 in the sense of $\mathcal{D}$

consequently $\lim_n T(\varphi_{x_n}) = T(\varphi_{\bar{x}})$

i.e. $\lim_n (T*\varphi)(x_n) = (T*\varphi)(\bar{x})$

similarly to prove that $T*\varphi$ is differentiable

$$\partial_{x_j}(T*\varphi)(\bar{x}) = \lim_{t\to 0} \frac{(T*\varphi)(\bar{x}+te_j) - (T*\varphi)(\bar{x})}{t} = \lim_{t\to 0} T\left(\frac{\varphi_{\bar{x}+te_j} - \varphi_{\bar{x}}}{t}\right)$$

$y \longmapsto \varphi_{\bar{x}+te_j}(y) = \varphi(\bar{x}+te_j - y)$

$\dfrac{\varphi_{\bar{x}+te_j} - \varphi_{\bar{x}}}{t} \longrightarrow \theta \qquad y \mapsto \theta(y) = \partial_{x_j}\varphi(\bar{x}-y)$

$\downarrow$

$\partial_{x_j}\varphi(\bar{x} - \cdot)$ in the sense of $\mathcal{D}$ .....

remark Let $(\varphi_n)_n$ be a sequence in $\mathcal{D}(\mathbb{R}^d)$

s.t. $\underbrace{\varphi_n \longrightarrow \varphi \text{ in the sense of } \mathcal{D}(\mathbb{R}^d)}$

Let $T \in \mathcal{D}'(\mathbb{R}^d)$

(supports in the same compact set
unif. convergence of $\partial^\alpha \varphi_n$ to $\partial^\alpha \varphi$ for all α)

~~then~~ $T * \varphi_n \longrightarrow T * \varphi$ in sense of $\underbrace{\mathcal{E}(\mathbb{R}^d)}$

(for every fixed K compact,
uniform convergence of
$\partial^\alpha(T * \varphi_n)$ to $\partial^\alpha(T*\varphi)$
for all α)

$\Updownarrow$

given a family of $\Omega_0 \subseteq \Omega_1 \subseteq \dots \subseteq \Omega_j \subseteq$
such that $\overline{\Omega_j} \subseteq \Omega_{j+1}$ and $\bigcup_j \Omega_j = \Omega$

($\overline{\Omega_j}$: compact; Ω_{j+1}: open)

there is the convergence for all the
seminorm $\tilde{p}_j(f) = \sum_{|\alpha| \le j} \sup_{x \in \Omega_j} |\partial^\alpha f(x)|$

I have to prove that, for all j fixed,

$$\lim_n \tilde{p}_j(T * \varphi_n - T * \varphi) = 0$$

$$\lim_n \sum_{|\alpha| \le j} \sup_{x \in \Omega_j} \left(\partial^\alpha (T * \varphi_n - T * \varphi)\right) = 0$$

it is not restrictive to consider $\varphi = 0$

$$\boxed{\lim_n \sum_{|\alpha|\le j} \sup_{x \in \Omega_j} |\partial^\alpha (T * \varphi_n)(x)| = 0 \quad ?}$$

compact

$$\partial^\alpha (T * \varphi_n) = T * \partial^\alpha \varphi_n$$
$$\parallel$$
$$T(\tilde{\varphi}_n)$$

$y \mapsto \partial^\alpha_x \varphi_n(x-y)$

considering $\tilde{K} = \overline{\Omega_j} - K$

T is distribution so $\exists\, m_{\tilde{K}},\ C_{\tilde{K}}$

s.t. $|T(\theta)| \le C_{\tilde{K}} \sum_{|\beta| \le m_{\tilde{K}}} \sup_{x \in \Omega} |\partial^\beta \theta|$

for all θ with supp in $\tilde{K}$

$$|T(\tilde{\varphi}_n)| \le \tilde{C}_{\tilde{K}} \sum_{|\alpha| \le m_{\tilde{K}} + j} \sup_{x \in \mathbb{R}^d} |\partial^\alpha \varphi_n|$$

Th. Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and let $\varphi, \psi \in \mathcal{D}(\mathbb{R}^d)$

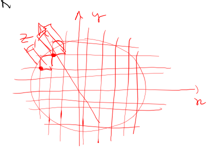
so $\quad T*(\varphi*\psi) = (T*\varphi)*\psi$

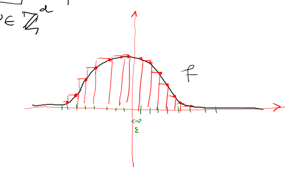
proof (sketch)

we remark that if $f \in \mathcal{C}_0(\mathbb{R}^d)$

continuous function with compact support.

$$\int_{\mathbb{R}^d} f(x)\,dx = \lim_{\varepsilon \to 0^+} \varepsilon^d \cdot \sum_{\nu \in \mathbb{Z}^d} f(\varepsilon\nu)$$





$$\varphi*\psi(x) = \int_{\mathbb{R}^d} \varphi(x-y)\,\psi(y)\,dy = \lim_{\varepsilon \to 0} \underbrace{\varepsilon^d \sum_{\nu \in \mathbb{Z}^d} \varphi(x-\varepsilon\nu)\,\psi(\varepsilon\nu)}_{f_\varepsilon(x)}$$

$$\varphi*\psi(x) = \lim_{\varepsilon \to 0^+} f_\varepsilon(x)$$

remark that $\quad x \longmapsto f_\varepsilon(x) \in \mathcal{C}_0^\infty = \mathcal{D}(\mathbb{R}^d)$

prove that $\quad f_\varepsilon \longrightarrow \varphi*\psi \quad$ in the sense of $\mathcal{D}$

so that $\quad T*(\varphi*\psi) = \lim_{\varepsilon \to 0^+} T*f_\varepsilon$

in particular $\quad T*(\varphi*\psi)(x) = \lim_{\varepsilon \to 0^+} (T*f_\varepsilon)(x)$

$$T*f_\varepsilon(x) = T*\Big(\varepsilon^d \sum_{\nu \in \mathbb{Z}^d} \psi(\varepsilon\nu)\,\varphi(x-\varepsilon\nu)\Big)$$

translation operator

$$\varphi(x-\varepsilon\nu) = \tau_{\varepsilon\nu}\varphi(x)$$

$$= \varepsilon^d \cdot \sum_{\nu \in \mathbb{Z}^d} \psi(\varepsilon\nu)\,(T*\tau_{\varepsilon\nu}\varphi)(x)$$

$$(T*\tau_{\varepsilon\nu}\varphi)(x) = T(\theta_{\nu,x})$$

where $\quad y \longmapsto \theta_{\nu,x}(y) = \varphi(x-\varepsilon\nu-y)$

$$= (T*\varphi)(x-\varepsilon\nu)$$

$$= \varepsilon^d \sum_{\nu \in \mathbb{Z}^d} (T*\varphi)(x-\varepsilon\nu)\cdot\psi(\varepsilon\nu)$$

and $\quad \displaystyle\int_{\mathbb{R}} (T*\varphi)(x-y)\cdot\psi(y)\,dy = \lim_{\varepsilon \to 0^+} \varepsilon^d \underbrace{\sum_{\nu \in \mathbb{Z}^d} (T*\varphi)(x-\varepsilon\nu)\,\psi(\varepsilon\nu)}$

$$T*(\varphi*\psi)(x) = \lim_{\varepsilon \to 0^+} T*f_\varepsilon(x) = \lim_{\varepsilon \to 0^+}$$

$$= \int_{\mathbb{R}^d} (T*\varphi)(x-y)\,\psi(y)\,dy = (T*\varphi)*\psi(x)$$

conclusion $\quad T*(\varphi*\psi)(x) = (T*\varphi)*\psi(x) \qquad$ QED

Another property of convolution of distr.

Th. let $T \in \mathcal{D}'(\mathbb{R}^d)$, let $\varphi \in \mathcal{D}(\mathbb{R}^d)$
let $h \in \mathbb{R}^d$

then $\tau_h(T * \varphi) = T * (\tau_h \varphi)$

(the convolution with T commutes with the translation)

proof. $\left(\tau_h(T * \varphi)\right)(x) = (T * \varphi)(x-h)$

$= T(\psi) \qquad y \mapsto \varphi(x-h-y)$

$= \left(T * (\tau_h \varphi)\right)(x) \qquad \|$

$= T(\theta) \qquad y \mapsto \tau_h \varphi(x-y)$

Theorem let $\Phi : \mathcal{D}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$

with i) Φ is linear

ii) Φ continuous (in the sense: if $(\varphi_n)_n$ converges to 0 in the sense of $\mathcal{D}$ then $\Phi(\varphi_n)$ converges to 0 in the sense of $\mathcal{E}$)

iii) Φ commutes with τ_h

then $\exists T \in \mathcal{D}'(\mathbb{R}^d)$ s.t. $\Phi(\varphi) = T * \varphi$
$\forall \varphi \in \mathcal{D}(\mathbb{R}^d)$

proof. I have Φ (I take $\varphi \in \mathcal{D}(\mathbb{R}^d)$, I consider $\check{\varphi}$, $\check{\varphi}(x) = \varphi(-x)$, $\check{\varphi} \in \mathcal{D}(\mathbb{R}^d)$)

I define $T(\varphi) = \Phi(\check{\varphi})(0)$ so that $\Phi(\check{\varphi}) \in \mathcal{E}^\infty$, I consider the value in 0

T is a distribution

T is linear $\quad T(\varphi_1 + \varphi_2) = \Phi((\varphi_1+\varphi_2)^\vee)(0) = \Phi(\check{\varphi}_1)(0) + \Phi(\check{\varphi}_2)(0)$

T is "continuous"

$(\varphi_n)_n \to 0$ in the sense of $\mathcal{D}$ then

$\Phi(\varphi_n) \to 0$ in the sense of $\mathcal{E}$

in particular $\Phi(\varphi_n)(0) \to 0$

I prove that $T * \varphi = \Phi(\varphi)$

$(T * \varphi)(x) = T(\psi_x) \qquad \psi_x(y) = \varphi(x-y)$

x parameter $\quad = \Phi(\check{\psi}_x)(0) \qquad \check{\psi}_x(y) = \varphi(x+y) = (\tau_{-x}\varphi)(y)$

$= \left(\Phi(\tau_{-x}\varphi)\right)(0) \qquad (\tau_{-x} f)(0) = f(x)$

$= \left(\tau_{-x}(\Phi(\varphi))\right)(0)$

$= \Phi(\varphi)(x)$

conclusion $\quad T * \varphi = \Phi(\varphi) \qquad$ **QED** def. Let $T \in \mathcal{D}'(\mathbb{R}^d)$, let $S \in \mathcal{E}'(\mathbb{R}^d)$

distribution

distribution with compact support

consider $\Phi : \mathcal{D}(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d)$

where $\Phi(\varphi) = T * (S * \varphi)$

remark that $S * \varphi \in \mathcal{E}_0^\infty$

compact support

now Φ is linear

Φ is continuous

Φ commutes with τ_h

$T * (S * \tau_h \varphi) =$

$T * (\tau_h (S * \varphi))$

$= \tau_h (T * (S * \varphi))$

so $\exists !\, W \in \mathcal{D}'(\mathbb{R}^d)$

s.t. $W * \varphi = T * (S * \varphi)$

I define $\boxed{W = T * S}$

$T * S = S * T$

Ex. 5-5.

$f \in L^1_{loc}(\mathbb{R}^d \setminus \{0\})$ and $\exists\, C > 0,\ \exists\, m \in \mathbb{N}$

s.t. $\quad |f(x)| \le \dfrac{C}{|x|^m}$

Prove that $\exists\, T \in \mathcal{D}'(\mathbb{R}^d)$ s.t. $\forall \varphi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$

$$T(\varphi) = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx$$

Solution $\qquad \varphi \in \mathcal{D}(\mathbb{R}^d)$

define $T(\varphi) =$

$$\frac{\cdot}{|x|^m} \quad C|x|^{m+1}$$

$$= \int_{|x|\ge 1} f(x)\varphi(x)\,dx + \int_{|x|\le 1} \underbrace{f(x)}_{L^1}\left(\varphi(x) - \sum_{|\alpha|\le m} \frac{\partial^\alpha \varphi(0)}{\alpha!}x^\alpha\right)dx$$

$\forall \varphi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ OK

$$\left|\varphi(x) - \sum_{|\alpha|\le m} \frac{\partial^\alpha \varphi(0)}{\alpha!}x^\alpha\right| \le C\,|x|^{m+1}$$